

# A System-theoretic Model for Cooperation, Interaction and Allocation Mechanisms

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## Abstract

A novel system theoretic approach to cooperation, interaction and allocation is presented that simplifies, unifies and extends the results on classical cooperative games and their generalizations. In particular, a general Weber theory of linear values is obtained and a new theory for local cooperation and general interaction indices is established. The model is dynamic and based on the notion of states of cooperation that change under actions of agents. Careful distinction between "local" states of cooperation and general "system" states leads to a notion of entropy for arbitrary non-negative and efficient allocations and thus to a new information theoretic criterion for fairness of allocation mechanisms. Shapley allocations, for instance, are exhibited as arising from random walks with maximal entropy. For a large class of cooperation systems, a characterization of game symmetries in terms of  $\lambda$ -values is given. A concept for cores and Weber sets is proposed and it is shown that Weber set of games with selection structure always contain the core.

*Keywords:* Allocation, cooperation, core, entropy, game, interaction, mechanism, randomization, state, symmetry, system, value, Weber set

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# 1 Introduction

The notions of cooperation, allocation and interaction of agents in specified environments have an intuitively dynamic flavor. Mathematical models for their analysis, however, are often surprisingly static and based on set theoretic frameworks where the formation of coalitions and allocations is described as the solution of discrete optimization problems with respect to given numerical parameters. The complexity of the model is thus seen mainly as the computational complexity of the resulting optimization problem. For example, an important part of current research in mechanism design contrasts the aspirations and the self-interest of individual agents with the common interest and investigates the algorithmic solution of the resulting optimization problem (see, *e.g.*, [12, 30, 31]). While these and other problems of computational complexity are non-trivial and interesting also from a game theoretic perspective (see, *e.g.*, [16, 17, 18]), they reflect only part of the picture.

The classical model of cooperative games assumes that arbitrary subsets of agents can join to form feasible coalitions and create values in a given economic context. A fundamental model for so-called probabilistic values that includes the Shapley value, the Banzhaf value *etc.* was developed for the classical model by Weber [38]. The assumption of unrestricted cooperation, however, does not always appear justified. Already the communication games of Myerson [32] exhibit restricted cooperation. The model of Kalai and Samet [24] is built on Aumann's games with block structure, where certain critical coalitions partition the set of agents. Hsiao and Raghavan [23] allow a player to choose participation in a game at various "action levels". The latter can be viewed as a cooperative game where the agents have to observe certain precedence constraints (Faigle and Kern [14, 15], Derks and Gilles [11]). Recently, Bilbao *et al.* have studied models for cooperative games with underlying combinatorial coalition structures (*cf.* [3]) such as convex geometries ([4, 6, 7]), antimatroids ([2]) or matroids ([5]). While the feasible coalitions in these models form lattices of sets, Lange and Grabisch [28] introduce a framework where the family of feasible coalitions is just assumed to enjoy a certain regularity condition. In all these generalizations, analogues of Shapley's [36] classical value (and possibly also the core) are sought.

Our present research wants to take a first step towards viewing cooperation, allocation and interaction as dynamic processes and thus to approach cooperative games system theoretically. We feel encouraged by the result as it appears to render the structural analysis both simpler and considerably

more general. Not only are all the above mentioned generalized models for cooperation and interaction easily identified as special cases. Also the classical results can be shown to extend to this wider context. The dynamic model furthermore suggests to study the entropy of a system of cooperation and relate it to fair allocations, which seems to offer a promising path of future research. The key in our analysis is a careful distinction between internal states of cooperation and general states of the system as a whole. The former states model evolving cooperation while the latter capture the likelihood of certain cooperation instances.

We introduce cooperation systems in Section 2 and review some basic properties. Section 3 discusses cooperative games in this framework and shows how the previous models fit into it. In Section 4, we develop a theory of efficient linear allocations and values and relate them to the notion of entropy of system states *via* natural random walks on the cooperation states. The Shapley allocation turns out to be characterized by having maximal entropy. Moreover, we extend Carreras and Owen’s [9] classical characterization of game symmetries in terms of weighted Shapley values to a large class of cooperation systems that includes, for example, all the above mentioned models (Theorem 4.5).

Interaction among agents is an important aspect of cooperative game theory that has received considerable attention by various researchers (*cf.* [33, 21, 27, 19]). We develop a new theory of general cooperation values and show in Section 5.2 how a general theory of interaction indices arises from it naturally. In the final Section 6, we introduce a model for the core and Weber sets in cooperation systems that arise from ranking models and establish the corresponding extension of Weber’s [38] theorem.

## 2 Cooperation Systems

A *cooperation system* is a quadruple  $\Gamma = (N, V, A, \mathcal{A})$ , where  $N$  is a finite set of *agents*,  $V$  a finite set of *states of cooperation* and  $A$  a finite set of *feasible transitions*  $x \rightarrow y$  between states, which we assume to be partitioned into pairwise disjoint blocks  $A_i$ , indexed by the agents  $i \in N$ . We denote the latter partition by  $\mathcal{A} = \{A_i \mid i \in N\}$  and think of the block  $A_i \in \mathcal{A}$  as the set of transitions that are governed by the agent  $i$ : intuitively,  $i$  can take the “action”  $(x \rightarrow y) \in A_i$  and transform the current state  $x$  of cooperation into the state  $y \in V$ .

Identifying a feasible transition  $(x \rightarrow y) \in A$  with the corresponding

pair  $xy \in V \times V$ , we obtain  $G = (V, A)$  as the (directed) *transition graph* of  $\Gamma$  with vertex set  $V$  and arc set  $A$ . For any  $x \in V$ , we set

$$x^- = \{u \in V \mid ux \in A\} \quad \text{and} \quad x^+ = \{y \in V \mid xy \in A\}.$$

$x$  is an *initial state* (or a *source*) of  $G$  if  $x^- = \emptyset$  and a *final state* (or *sink*) if  $x^+ = \emptyset$ . In the case  $x^- \neq \emptyset \neq x^+$ ,  $x$  is a *transitory state* (or *inner vertex*). For simplicity of exposition, we assume throughout

( $\Gamma_0$ ) There is one unique initial state  $s \in V$ .

( $\Gamma_1$ )  $G = (V, A)$  is acyclic (*i.e.*,  $G$  does not contain any directed cycle).

By ( $\Gamma_0$ ) and ( $\Gamma_1$ ), every state of cooperation  $x \in V$  can be reached *via* a directed path from  $s$  to  $x$  in  $G$ . Moreover, each path extends to a path that ends in a sink  $t$ . Let  $T$  be the collection of all sinks  $t$ . We denote by  $\mathcal{P}_t$  the family of all paths  $P = sx_1 \dots x_k t$  from the source  $s$  to the sink  $t \in T$  and thus obtain the family of all source-sink paths in  $G$  as

$$\mathcal{P} = \bigcup_{t \in T} \mathcal{P}_t.$$

A *phase of cooperation* is a pair  $(x, xy)$  with  $x \in V$  and  $xy \in A$ . So the members of  $\mathcal{P}$  correspond to the trajectories from the initial state  $s$  to final states  $t$  in the cooperation space.

The cooperation system  $\Gamma = (N, G, \mathcal{A})$  could be regarded as the *extensive form* of cooperation involving the agents in  $N$ , where the agents sequentially change the state of cooperation until a final state is reached. A trajectory  $P \in \mathcal{P}$  is therefore called a *cooperation instance*.

## 2.1 System States and Entropy

We distinguish between the states of cooperation of  $\Gamma$ , *i.e.*, the vertices of the graph  $G = (V, A)$ , and the states of system  $\Gamma = (N, G, \mathcal{A})$ . We take the cooperation instances of (*i.e.*, the source-sink paths  $P$ ) as the *pure states* of the system  $\Gamma$  and define a (general) *mixed state* as a (formal) convex combination (superposition)

$$\rho = \sum_{P \in \mathcal{P}} \pi_P P \quad (\text{with } \pi_P \geq 0 \text{ and } \sum_{P \in \mathcal{P}} \pi_P = 1) \quad (1)$$

of pure states. Interpreting the coefficient vector  $\pi = (\pi_P \mid P \in \mathcal{P})$  in (1) as a probability distribution on the collection  $\mathcal{P}$  of cooperation instances, it is

convenient to identify the mixed state  $\rho$  with the corresponding probability distribution  $\pi = \pi(\rho)$  on  $\mathcal{P}$ .

TERMINOLOGY. For a clearer exposition, we will subsequently often refer to cooperation states  $x$  simply as "states" and to system states  $\rho$  (indirectly) as "probability distributions"  $\pi = \pi(\rho)$ .

### 2.1.1 Localized Systems

Consider a fixed cooperation state  $x_0 \in V$  and let  $V(x_0)$  be the set of all  $x \in V$  that can be reached by directed paths from  $x_0$ . Then  $x_0$  is the source of the *localized* cooperation system

$$\Gamma(x_0) = (N, V(x_0), A(x_0), \mathcal{A}(x_0)),$$

where  $A(x_0)$  and  $\mathcal{A}(x_0)$  are the analogous restrictions of  $A$  and  $\mathcal{A}$ .

Assume that  $\Gamma$  is in the system state  $\rho$  with probability distribution  $\pi = \pi(\rho)$ . Let  $P$  be a path from  $x_0$  to some sink. Denoting by  $\mathcal{P}_{x_0}$  the collection of all paths from the source  $s$  of  $\Gamma$  to  $x_0$ ,  $P$  is realized in  $\Gamma(x_0)$  with probability

$$\pi_P^0 = \sum_{P' \in \mathcal{P}_{x_0}} \pi_{P'P},$$

where  $P'P$  is the concatenation of the paths  $P'$  and  $P$  at  $x_0$ . So we find the localized system  $\Gamma(x_0)$  to be in the well-defined system state  $\rho^0$  with associated probability distribution  $\pi^0 = \pi(\rho^0)$ .

### 2.1.2 Entropy

The *entropy*  $H(\rho)$  of the system state  $\rho$  of  $\Gamma$  is the parameter

$$0 \leq H(\rho) = - \sum_{P \in \mathcal{P}} \pi_P(\rho) \log_2 \pi_P(\rho) \leq \log_2 |\mathcal{P}|. \quad (2)$$

$H(\rho) = 0$  is equivalent with  $\rho$  being a pure state.  $H(\rho) = \log_2 |\mathcal{P}|$  means that all cooperation instances are equally likely to be realized.

## 2.2 Flows and Potentials

A *flow* in the cooperation system  $\Gamma$  is a (formal) linear combination

$$f = \sum_{xy \in A} f_{xy}(x \rightarrow y) \quad (3)$$

of state transitions. For simplicity, we identify a flow with its coefficient vector, *i.e.*, we set

$$f = (f_{xy} \mid xy \in A) \in \mathbb{R}^A$$

and use following the notation for the flow *into* (resp. *out of*) a state  $x \in V$ :

$$f^-(x) = \sum_{u \in x^-} f_{ux} \quad \text{and} \quad f^+(x) = \sum_{y \in x^+} f_{xy}.$$

We say that  $f \in \mathbb{R}^A$  is an *s-flow* if  $f$  satisfies the conservation law

$$f^-(x) = f^+(x) \quad \text{for each transitory state } x \in V. \quad (4)$$

Let  $P = sx_1 \dots x_k t \in \mathcal{P}$  be the trajectory that passes through the vertices  $s, x_1, \dots, x_k, t$  of  $G$  along the arcs  $sx_1, x_1x_2, \dots, x_k t$ . We may think of  $P$  as the *elementary flow*  $f(P) \in \mathbb{R}^A$  with components

$$f_{xy}(P) = \begin{cases} 1 & \text{if } xy \in P \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that  $f(P)$  is an *s-flow* for every  $P \in \mathcal{P}$ . In fact, the following is well-known:

**Lemma 2.1**  *$f \in \mathbb{R}^A$  is an s-flow if and only if there are parameters  $f_P \in \mathbb{R}$  such that*

$$f = \sum_{P \in \mathcal{P}} f_P f(P).$$

◇

A *potential* on  $\Gamma$  is a (formal) linear combination  $v$  of (cooperation) states, which we identify with a coefficient vector in  $\mathbb{R}^V$ :

$$v = \sum_{x \in V} v_x x \quad \longleftrightarrow \quad v = (v_x \mid x \in V) \in \mathbb{R}^V. \quad (5)$$

A constant potential  $v \equiv \text{const}$  is thought to produce no effect in the system  $\Gamma$ . So we consider two potentials  $v, w \in \mathbb{R}^V$  to be *equivalent* if  $v - w \equiv \text{const}$ . This is conveniently expressed with the *marginal operator* (or *boundary operator*)  $\partial : \mathbb{R}^V \rightarrow \mathbb{R}^A$ , where

$$\partial_{xy}(v) = v(y) - v(x) \quad \text{for all } xy \in A,$$

associates with the potential  $v$  its *marginal flow*  $f = \partial(v)$ .

**Lemma 2.2**  $v, w \in \mathbb{R}^V$  are equivalent if and only if  $\partial(v) = \partial(w)$ .

◇

Extending the marginal operator to arbitrary paths  $P = x_0x_1 \dots x_k$  in the transition graph  $G = (V, A)$ , we set

$$\partial_P(v) = \partial_{x_0x_1}(v) + \dots + \partial_{x_{k-1}x_k}(v) = v(x_k) - v(x_0).$$

It follows that the potential  $v$  is uniquely determined by  $\partial(v)$  and its value  $v(s)$  on the initial state  $s \in V$ . Indeed, we have

$$v(x) = v(s) + \partial_P(v) \quad \text{for all paths } P = s \dots x \text{ in } G.$$

So we may restrict our attention to *zero-normalized* potentials  $\bar{v}$  with values  $\bar{v}(x) = v(x) - v(s)$  (and consequently  $\bar{v}(s) = 0$ ) and henceforth use the term "potential" to mean "zero-normalized potential".

Given a flow  $f \in \mathbb{R}^A$  and a path  $P = x_0x_1 \dots x_kx$ , we define the corresponding (discrete) *path integral*:

$$\int_P f = f_{x_0x_1} + \dots + f_{x_{k-1}x_k}. \quad (6)$$

**Lemma 2.3** Let  $f \in \mathbb{R}^A$  be a flow. Then  $f = \partial(v)$  holds for some potential  $v$  if and only if for every sink  $t \in T$  and any two paths  $P, P' \in \mathcal{P}_t$  the path integrals coincide:

$$\int_P f = \int_{P'} f.$$

By (6),  $v$  is uniquely determined by  $f$ . Hence the  $(|V| - 1)$ -dimensional vector space  $\mathcal{V} \subseteq \mathbb{R}^V$  of zero-normalized potentials is isomorphic with the vector space  $\partial(\mathcal{V}) \subseteq \mathbb{R}^A$  of  $s$ -flows.

◇

**Simple Potentials.** Relative to any  $x \in V$ , we define the *simple potential*  $\zeta_x : V \rightarrow \mathbb{R}$  via

$$\zeta_x(y) = \begin{cases} 1 & y \text{ lies on a path from } s \text{ to } x \text{ in } G \\ 0 & \text{otherwise.} \end{cases}$$

The simple potentials form a basis of the vector space of all potentials:

**Lemma 2.4** For every  $v \in \mathcal{V}$ , there exist uniquely determined coefficients  $\beta_x(v)$  such that

$$v = \sum_{x \in V} \beta_x(v) \zeta_x.$$

*Proof.* Consider the  $(0, 1)$ -matrix  $Z = [\zeta_{xy}] \in \{0, 1\}^{V \times V}$  with coefficients  $\zeta_{xy} = \zeta_x(y)$ . It is not hard to see that the column and row indices  $x, y$  can be arranged so that  $Z$  is lower triangular with diagonal elements  $\zeta_{xx} = 1$ . Hence  $Z$  has full rank  $|V|$ , i.e., every potential  $v \in \mathbb{R}^V$  as a unique representation as a linear combination of the rows of  $Z$ . ◇

Notice that the coefficients  $\beta_x(v)$  are linear in  $v$ :

$$\begin{aligned} \sum_{x \in V} \beta_x(v + w) \zeta_x &= v + w = \sum_{x \in V} [\beta_x(v) + \beta_x(w)] \zeta_x \\ \sum_{x \in V} \lambda \beta_x(v) \zeta_x &= \lambda v = \sum_{x \in V} \beta_x(\lambda v) \zeta_x. \end{aligned}$$

Moreover, if  $v$  is zero-normalized, we must have  $\beta_s(v) = 0$ , i.e.,

$$v = \sum_{x \neq s} \beta_x(v) \zeta_x \tag{7}$$

REMARK. Lemma 2.4 can be viewed within the general framework of Rota's [34] so-called incidence algebra. (See, e.g., Stanley [37] for more details.)

### 3 Cooperative Games

We now show how the models of classical cooperative games and their generalizations fit into the framework of cooperation systems  $\Gamma = (N, V, A, \mathcal{A})$ .  $\Gamma$  captures all the possible cooperation instances. The analysis of cooperation that might (or should) take place, however, depends on further parameters. For example,

- an assessment (or *valuation*) by a real parameter  $v(x) \in \mathbb{R}$  of the state  $x \in V$  of cooperation;
- an assessment  $f_{xy} \in \mathbb{R}$  agent  $i \in N$  may have regarding the value of his action  $xy \in A_i$  and a corresponding reward expectation;

- an assessment  $\pi_{xy} \in [0, 1]$  of the probability that (cooperation) state  $x$  is changed into state  $y$  in a randomized cooperation instance.

We define a *cooperative game* to be a pair  $(\Gamma, v)$ , where  $v : V \rightarrow \mathbb{R}$  is a valuation of the states of cooperation.  $v$  is the so-called *characteristic function* of the game  $(\Gamma, v)$ . By slight abuse of language, one sometimes refers to  $v$  itself as a "cooperative game". As usual, we make the assumption that characteristic functions  $v$  are *zero-normalized* in the sense

$$(ZN) \quad v(s) = 0.$$

So the vector space  $\mathcal{V}$  of all characteristic functions (relative to  $\Gamma$ ) coincides with the space of (zero-normalized) potential on  $\Gamma$ . Next, we present a generic set theoretic model that incorporates all the classical cooperative games and their generalizations in the literature relative to a set  $N$  of agents.

**Selectors.** A *selector* is an operator  $X \mapsto \sigma(X)$  on the subsets (a.k.a. *coalitions*) of  $N$  such that

$$\sigma(X) \subseteq N \setminus X \quad \text{for all } X \subseteq N.$$

A *ordered selection* (a.k.a. *ranking*) is a sequence  $\pi = p_1 \dots p_k$  of agents with the property

$$p_i \in \sigma(\{p_1, \dots, p_{i-1}\}) \quad (i = 1, \dots, k).$$

The underlying set  $S = \{p_1, \dots, p_k\}$  of the ranking  $\pi$  is a *selection*.

Let  $\mathcal{S}$  be the collection of all selections. In accord with the terminology of combinatorial structure theory, we refer to a selection  $B \in \mathcal{S}$  with  $\sigma(B) = \emptyset$  as a *basis* (of  $\sigma$ ).  $\mathcal{B}$  denotes the collection of all bases.

**Example 3.1** A *greedoid* (see [25]) is a pair  $(N, \sigma)$ , where  $\sigma$  is a selector so that for all selections  $S, S' \in \mathcal{S}$  the so-called Steinitz exchange property is guaranteed:

$$(G) \quad |S| < |S'| \implies \sigma(S) \cap S' \neq \emptyset.$$

Observe that all the generalizations of classical cooperative games investigated in [1, 2, 5, 4, 6, 7, 11, 14, 28, 23, 24] yield special cases of greedoids. A *regular set system* in the sense of Lange and Grabisch [28] arises not necessarily from a greedoid, but from a selector with the unique basis property  $\mathcal{B} = \{N\}$ .

REMARK. The concept of a selector is dual to that of a *choice function*, namely an operator  $X \mapsto \gamma(X)$  such that  $\gamma(X) \subseteq X$  (see, e.g., [10, 26, 29]).

**Cooperative Games with Selection Structure.** Associate with the selector  $\sigma$  the cooperation system  $\Gamma(\sigma)$  with the selections  $X \in \mathcal{S}$  as cooperation states and the individual action sets

$$A_p = \{(X, X \cup p) \mid X \in \mathcal{S}, p \in \sigma(X)\} \quad (p \in N).$$

So the empty set  $\emptyset$  is the initial state of  $\Gamma(\sigma)$  and the final states are precisely the  $\sigma$ -bases. Moreover, a path  $P$  from  $\emptyset$  to some  $X \in \mathcal{S}$  in the transition graph  $G = (\mathcal{S}, A)$  corresponds to a ranking  $\pi = p_1, \dots, p_k$  of the members of  $X$ .

The pair  $(\Gamma(\sigma), v)$  is a *cooperative game with coalition structure*  $(N, \sigma)$ .

**Example 3.2 (Full Cooperation)** Let  $\sigma_0$  be the selector given by

$$\sigma_0(X) = N \setminus X \quad \text{for all } X \subseteq N.$$

The associated set  $\mathcal{S}_0$  of selections comprises all subsets  $X \subseteq N$ . Hence we refer to  $\Gamma(\sigma_0)$  as the *full cooperation system*. A cooperative game of the type  $(\Gamma(\sigma_0), v)$  is called *classical* and is specified by the (zero-normalized) function  $v : \mathcal{S}_0 \rightarrow \mathbb{R}$ . A classical agent  $p \in N$  governs the action set

$$A_p = \{(X, X \cup p) \mid X \subseteq N \setminus p\}.$$

Notice that all cooperative games with selector structure have the following *singular action property*

(SA)  $|P \cap A_i| \leq 1$  holds for all cooperation instances  $P \in \mathcal{P}$ , i.e., any agent  $i \in N$  acts at most once.

## 4 Allocations and Values

Given the arbitrary cooperation system  $\Gamma = (N, V, A, \mathcal{A})$ , we define an *allocation mechanism* to be a computational scheme for allocating payoffs to the individual agents  $i \in N$  in the context of a cooperative game  $(\Gamma, v)$ . We make the following axiomatic assumptions:

- (A<sub>0</sub>) The null game  $v \equiv 0$  should yield zero payoffs.
- (A<sub>1</sub>) The allocation to any agent  $i \in N$  should only depend on his action set  $A_i \in \mathcal{A}$  and should be linear in  $v$ .

( $A_0$ ) and ( $A_1$ ) imply that equivalent potentials produce the same allocations. In other words, the allocation relative to the potential  $v$  depends only on the marginal flow  $\partial(v)$ . Since the vector space  $\mathcal{V}$  of cooperative games is isomorphic to the vector space  $\partial(\mathcal{V})$  of marginal flows, a linear allocation mechanism is therefore described by a parameter vector  $\alpha \in \mathbb{R}^A$  that determines the individual payoffs (or *individual values*)

$$\varphi_i^\alpha(v) = \sum_{xy \in A_i} \alpha_{xy} \partial_{xy}(v) = \sum_{xy \in A_i} \alpha_{xy} [v(y) - v(x)] \quad (8)$$

for the agents  $i \in N$ . We call the linear functional

$$v \mapsto \varphi^\alpha(v) = \alpha^T \partial(v) = \sum_{xy \in A} \alpha_{xy} \partial_{xy}(v) = \sum_{i \in N} \varphi_i^\alpha(v) \quad (9)$$

the (*group*) *value* associated with the allocation mechanism  $\alpha$ .

## 4.1 Efficient Allocations

Let  $\alpha \in \mathbb{R}^A$  be an allocation mechanism. We say that  $\alpha$  is *efficient* if the associated group value  $v \mapsto \varphi^\alpha(v)$  enjoys the following property:

(E) There are parameters  $\mu_t \in \mathbb{R}$  such that

$$\sum_{t \in T} \mu_t = 1 \quad \text{and} \quad \varphi^\alpha(v) = \sum_{t \in T} \mu_t v(t) \quad \text{for all } v \in \mathcal{V}.$$

**Example 4.1** A group value  $\varphi$  for classical cooperative games is called *efficient* if  $\varphi(v) = v(N)$  holds for all  $v$ . Since  $t = N$  is the unique sink in the full cooperation system  $\Gamma(\sigma_0)$ , this notion agrees with our general definition.

**Theorem 4.1** The allocation mechanism  $\alpha \in \mathbb{R}^A$  is efficient if and only if  $\alpha$  is an  $s$ -flow with the property  $\alpha^+(s) = 1$ .

*Proof.* Define the Dirac potential  $\delta_x : V \rightarrow \{0, 1\}$  for any  $x \neq s$  by

$$\delta_x(y) = 1 \quad \iff \quad y = x.$$

Then we have  $v = \sum_{x \neq s} v(x) \delta_x$  for every  $v \in \mathcal{V}$ . Moreover, any  $\alpha \in \mathbb{R}^A$  yields

$$\varphi^\alpha(\delta_x) = \sum_{wz \in A} \alpha_{wz} [\delta_x(z) - \delta_x(w)] = \alpha^-(x) - \alpha^+(x).$$

If  $\alpha$  is efficient, we have  $\varphi^\alpha(\delta_x) = 0$  for every transitory state  $x$  and conclude that  $\alpha$  must be an  $s$ -flow. Moreover, for the potential  $\delta = \sum_{t \in T} \delta_t$  we find

$$1 = \sum_{t \in T} \mu_T \cdot 1 = \varphi^\alpha(\delta) = \sum_{t \in T} \sum_{yt \in A} \alpha_{yt} = \sum_{sx \in A} \alpha_{sx} = \alpha^+(s).$$

Conversely, if  $\alpha$  is an  $s$ -flow, we have by linearity

$$\varphi^\alpha(v) = \varphi^\alpha\left(\sum_{x \neq z} v(x)\delta_x\right) = \sum_{t \in T} v(t)\varphi^\alpha(\delta_t).$$

Furthermore, we observe

$$\alpha^+(s) = 1 \implies \sum_{t \in T} \varphi^\alpha(\delta_t) = \sum_{t \in T} \alpha^-(t) = \alpha^+(s) = 1.$$

So the claim of the Theorem follows with  $\mu_t = \varphi^\alpha(\delta_t)$ . ◇

We now proceed to show that non-negative efficient allocation mechanisms correspond to system(!) states of  $\Gamma$  and thus have a well-defined entropy.

## 4.2 Randomization

Let  $\alpha \in \mathbb{R}^A$  be an efficient allocation mechanism with non-negative components  $\alpha_{xy} \geq 0$ . We associate with  $\alpha$  a random walk on the transition graph  $G = (V, A)$  of the cooperation system  $\Gamma$  according to the following rules:

- ( $R_0$ ) The walk starts in the initial state  $s$ .
- ( $R_1$ ) The walk moves from the state  $x$  to the state  $y$  along the arc  $xy \in A$  with probability  $\pi_{xy} = \alpha_{xy}/\alpha^+(x)$  if  $x$  is not a final state. If  $x$  is a final state, the random walk stops.

Notice that the random walk is well-defined: a state  $x \neq s$  can only be reached if  $\alpha^-(x) > 0$  holds. But then also  $\alpha^+(x) > 0$  is guaranteed if  $x$  is a transitory state. So the random walk stops eventually in some final state  $t \in T$ , having traversed some source-sink path  $P \in \mathcal{P}$ .

Denote by  $\mathcal{P}_x$  the family of all paths from  $s$  to the state  $x$  and let  $\pi(P_x)$  be the probability that the random walk follows initially the path  $P_x = s \dots x \in \mathcal{P}_x$ . Then the probabilities  $\pi(P_x)$  can be recursively computed according to the rule

$$\pi(s \dots xy) = \begin{cases} 0 & \text{if } \alpha^+(x) = 0 \\ \pi(s \dots x)\pi_{xy} & \text{if } \alpha^+(x) > 0. \end{cases}$$

The crucial observation is the following:

**Lemma 4.1** *Let  $\alpha \in \mathbb{R}_+^A$  be a non-negative and efficient allocation mechanism. Then each component  $\alpha_{xy}$  of  $\alpha$  equals the probability  $Pr(x, y)$  that the random walk associated with  $\alpha$  passes through the arc  $xy \in A$ .*

*Proof.* Without loss of generality, we may assume  $\alpha_{xy} > 0$  for all  $xy \in A$ . Since the transition graph  $G$  is acyclic, we can order the vertices  $x_0, x_1, \dots, x_m$  of  $G$  such that we have for all indices  $0 \leq i, j \leq m$ :

$$x_i x_j \in A \implies i < j.$$

Since  $s$  is the unique source of  $G$ , we have  $x_0 = s$ . In view of  $\alpha^+(s) = 1$ , the statement of the Lemma is clearly true for all arcs of the form  $x_0 y = s y$ . Assume now that  $Pr(u, z) = \alpha_{uz}$  has been established for all arcs  $uz$  with  $u = x_i$  for some  $i \leq k - 1$  and consider the arc  $x_k y$ . Then one finds

$$\begin{aligned} Pr(x_k, y) &= \sum_{u \in x_k^-} Pr(u, x_k) \pi_{x_k y} \\ &= \sum_{u \in x_k^-} \alpha_{u, x_k} \pi_{x_k y} \\ &= \alpha^-(x_k) \pi_{x_k y} = \alpha^+(x_k) \pi_{x_k y} = \alpha_{x_k y}. \end{aligned}$$

◇

Lemma 4.1 offers a stochastic interpretation of non-negative efficient linear values  $\varphi^\alpha$ :

- Perform an  $\alpha$ -random walk on  $G = (V, A)$  and allocate to the owner of the arc  $xy \in A$  the payoff  $\partial_{xy}(v) = v(y) - v(x)$  if the random walk passes through  $xy$ .
- The individual value  $\varphi_i^\alpha$  of agent  $i \in N$  is exactly his expected payoff total:

$$\varphi_i^\alpha(v) = \sum_{xy \in A_i} \alpha_{xy} \partial_{xy}(v).$$

**Entropy and Fair Allocation.** The  $\alpha$ -random walk according to  $(R_0)$  and  $(R_1)$  generates a probability distribution  $\pi$  on the family  $\mathcal{P}$  of all cooperation instances. Moreover, Lemma 4.1 shows that  $\alpha$  is uniquely determined by  $\pi$ . So  $\alpha$  defines a unique system state  $\rho(\alpha)$ . Hence it is meaningful to

define the *entropy*  $H(\alpha)$  of the non-negative efficient allocation mechanism  $\alpha$  as

$$H(\alpha) = H(\rho(\alpha)) = - \sum_{P \in \mathcal{P}} \pi_P \log_2 \pi_P . \quad (10)$$

The notion of entropy suggests to consider an allocation mechanism  $\alpha$  for the be *fair* if

$$H(\alpha) = H(\rho)$$

is true, where  $\rho$  is the presumed system state of  $\Gamma$ .

**REMARK.** Note that the entropy  $H(\alpha)$  refers to a particular allocation mechanism  $\alpha$  for the system  $\Gamma$  and not a particular game  $(\Gamma, v)$ . For an entropy notion for set functions  $v$ , see Honda and Grabisch [22].

### 4.3 Random Values

We now assume an arbitrary probability distribution  $\pi$  on the cooperation instances  $P \in \mathcal{P}$  to be given.  $\pi$  induces an allocation mechanism  $\alpha \in \mathbb{R}^A$  via

$$\alpha_{xy} = \sum_{P \ni xy} \pi_P .$$

The group value  $v \mapsto \varphi^\alpha(v)$  is called the *random value* induced by  $\pi$ .

**Lemma 4.2** *The allocation mechanism  $\alpha$  induced by the probability distribution  $\pi$  on  $\mathcal{P}$  is non-negative and efficient.*

*Proof.*  $\alpha_{xy} \geq 0$  follows from  $\pi_P \geq 0$  for all  $P \in \mathcal{P}$ .  $\alpha^-(x) = \alpha^+(x)$  for all transitory states  $x$  is a consequence of the fact that a random path enters  $x$  and leaves  $x$  with the same probability. Finally, one observes

$$\alpha^+(s) = \sum_{x \in s^+} \sum_{P \ni sx} \pi_P = \sum_{P \in \mathcal{P}} \pi_P = 1.$$

◇

From Lemma 4.1 and Lemma 4.2 we infer:

**Theorem 4.2** *The class of non-negative efficient allocation mechanisms relative to the cooperation system  $\Gamma = (N, V, A, \mathcal{A})$  corresponds to the class of system states of  $\Gamma$ .*

◇

## 4.4 Shapley Allocations

We assume to be given non-negative parameters  $\mu_t \geq 0$  such that  $\sum_{t \in T} \mu_t = 1$  (i.e. a probability distribution  $\mu$  on the set  $T$  of final states) and consider the linear functional

$$v \mapsto \varphi(v) = \sum_{t \in T} \mu_t v(t)$$

on the vector space  $\mathcal{V}$  of (zero-normalized) potentials. We seek a non-negative efficient allocation mechanism  $\alpha \in \mathbb{R}^A$  with the property  $\varphi = \varphi^\alpha$ . Such a mechanism is easily obtained. We define a probability distribution  $\pi^S$  on  $\mathcal{P}$  via

$$\pi_P^S = \frac{\mu_t}{|\mathcal{P}_t|} \quad \text{if } P \in \mathcal{P}_t$$

and let  $\alpha^S$  be the associated mechanism. We call  $\alpha^S$  the *Shapley allocation mechanism* relative to  $\mu = \{\mu_t \mid t \in T\}$ . Then one finds

**Theorem 4.3** *The Shapley allocation mechanism  $\alpha^S$  is the unique non-negative and efficient allocation mechanism  $\alpha$  of maximal entropy relative to the property  $\varphi^\alpha = \varphi$ .*

◇

### 4.4.1 Shapley Values

We let  $\Phi^\mu$  be the group value induced by the Shapley allocation mechanism  $\alpha^S$  above and call it the *Shapley value* (relative to the parameter set  $\mu$ ).

In the full cooperation system  $\Gamma(\sigma_0)$ , the so-called *grand coalition*  $N$  is the unique final state. So  $v \mapsto v(N)$  corresponds to a linear functional  $\varphi$  with  $\mu = \{1\}$ . In this case, the Shapley mechanism  $\alpha^S$  yields exactly the classical allocation scheme introduced by Shapley [36], which has become known as the *Shapley value*  $\Phi = \Phi^{(1)}$ .

A similar situation occurs in any cooperation system  $\Gamma(\sigma)$  arising from a selector  $\sigma$  with the unique basis  $N$ . Here, our Shapley value yields the *Kirchhoff Shapley value* of Lange and Grabisch [28]. The "Shapley value" of Faigle and Kern [14] for precedence orders corresponds to the Shapley mechanism of the associated ranking structure.

The "Shapley value" for cooperative games on a matroid with collection  $\mathcal{B}$  of bases proposed by Bilbao *et al.* [5] is the Shapley value  $\Phi^\mu$  relative to the linear functional

$$\varphi(v) = \sum_{B \in \mathcal{B}} \mu_B v(B) \quad \text{with} \quad \mu_B \geq 0, \quad \sum_{B \in \mathcal{B}} \mu_B = 1.$$

#### 4.4.2 Myerson and Position Value

The *Myerson value* discussed in Algaba *et al.* [1] for a generalization of Myerson's [32] communication games is the Shapley value applied to a special class of classical cooperative games. The *position value* proposed in Borm *et al.* [8] for Myerson games also arises essentially from the Shapley value. The underlying model, however, is more general:

One starts from a partition  $N = N_1 \cup \dots \cup N_m$  of the agents into pairwise disjoint and non-empty groups  $N_i$  and thinks of the  $N_i$  as  $m$  individually acting "players". The cooperative game is assumed to be defined relative to  $N$ . However, one seeks to divide the value  $v(N)$  not among the  $|N|$  agents but among the  $m$  players  $N_i$ . The *position value* now computes the Shapley value relative to  $N$  and then assigns to  $N_i$  the sum of the Shapley allocations of the agents in  $N_i$ . The latter model amounts to the choice

$$A_i = \{(X, X \cup p) \mid p \in N_i \setminus X\}$$

as the arc set owned by  $N_i$ . So the position value is a special case of the general Shapley allocation mechanism.

#### 4.5 $\lambda$ -Mechanisms

A parameter vector  $\lambda \in \mathbb{R}^N$  with positive components  $\lambda_i > 0$  defines a random walk on the transition graph  $G$  that starts with the initial state  $s$  and chooses a successor of the current state  $x$  according to the rule

( $L_0$ ) Assign weight  $\lambda_{xy} = \lambda_i$  to the arcs  $xy \in A_i$  for all  $i \in N$ .

( $L_1$ ) Move from  $x$  to  $y$  with probability  $\pi_{xy} = \lambda_{xy} / \lambda^+(x)$ .

We denote the induced allocation mechanism by  $\alpha^\lambda$  and the associated group value by  $\Psi^\lambda$ .  $\Psi^\lambda$  is the so-called  $\lambda$ -value.

Assume that the transition graph  $G = (V, A)$  is *regular* in the sense that  $V$  can be partitioned into  $V = V_0 \cup V_1 \cup \dots \cup V_k$  such that

( $V_0$ ) If  $xy$  is an arc with  $x \in V_j$ , then  $y \in V_{j+1}$ .

( $V_1$ ) All vertices  $x$  in a block  $V_j$  have the same outdegree  $\Delta^+(x)$ .

Then one obtains the Shapley value as a special  $\lambda$ -value:

**Proposition 4.1** *If  $G$  is regular, then  $\Phi^{(1)} = \Psi^{(1, \dots, 1)}$ .*

◇

REMARK. It is not difficult to see that the conclusion of Proposition 4.1 may be false if  $G$  is not regular.

### 4.5.1 The Weighted Shapley Value

Assume the set  $N = N_1 \cup \dots \cup N_m$  of agents partitioned into the pairwise disjoint and non-empty blocks  $N_j$ . For each  $X \subseteq N$ , let  $j(X)$  be the largest index  $j$  such that  $X \cap N_j \neq \emptyset$  and consider the system  $\Gamma(\sigma)$  with

$$\sigma(X) = \begin{cases} N_{j(X)} \setminus X & \text{if } N_{j(X)} \setminus X \neq \emptyset \\ N_{j(X)+1} & \text{otherwise.} \end{cases}$$

$\Gamma(\sigma)$  has a regular transition graph  $G(\sigma)$ . Moreover, the associated  $\lambda$ -value  $\Psi^\lambda$  is precisely the  $\lambda$ -weighted Shapley value of Kalai and Samet [24]. In the case of the trivial partition  $N = N_1$  with just one block, we obtain the classical model of cooperative games. In particular,  $\Psi^\lambda$  becomes the generalized value of Shapley [35].

**Example 4.2** Let  $S \subseteq N$  be a fixed non-empty set of agents in the Kalai-Samet model and consider the cooperative game with potential

$$v_S(X) = \begin{cases} 1 & \text{if } X \cap S \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Given  $\lambda > 0$ , a random cooperation instance will assign the marginal value 1 to the first agent of  $S$  becoming active. This is necessarily an agent in  $S_{\min}$ . The probability for  $i \in S_{\min}$  to be the first selected is

$$\Psi_i^\lambda(v_S) = \frac{\lambda_i}{\sum_{\ell \in S_{\min}} \lambda_\ell}. \quad (11)$$

In other words:  $\Psi^\lambda$  distributes the value  $v_S(N) = 1$  among the  $i \in S_{\min}$  in proportion to their respective weight  $\lambda_i$ .

## 4.6 Monotone Potentials and Symmetries

We say that the potential  $v$  of  $\Gamma = (N, V, A, \mathcal{A})$  is *monotone (increasing)* if for all  $x, y \in V$ ,

$$xy \in A \implies v(x) \leq v(y).$$

In order to characterize monotone potentials by non-negative individual pay-offs, we assume  $\Gamma$  to have the singular action property

(SA)  $|P \cap A_i| \leq 1$  for all  $P \in \mathcal{P}$  and  $i \in N$ .

Recall from Section 3 that all cooperative games with selector structure enjoy property (SA), for example.

**Theorem 4.4** *Assume that (SA) holds for  $\Gamma$ . Then a potential  $v$  is monotone if and only if every  $\lambda$ -value  $\Psi^\lambda$  yields non-negative individual payoffs*

$$\Psi_i^\lambda(v) \geq 0 \quad \text{for all } i \in N.$$

*Proof.* Since random allocation mechanisms are non-negative, a monotone potential  $v$  will guarantee non-negative individual payoffs  $\varphi_i^\alpha(v) \geq 0$  for any random value  $\varphi^\alpha$  and hence for  $\lambda$ -values in particular.

Conversely, suppose that  $v(x) < v(y)$  holds for some arc  $xy \in A_k$ . Choose a path  $P = x_0x_1 \dots x_m \in \mathcal{P}$  that passes through  $xy$  and assume  $x_{i-1}x_i \in A_i$  for  $i = 1, \dots, m$  and  $xy = x_{k-1}x_k$  in particular. Consider a positive parameter vector  $\lambda \in \mathbb{R}^N$  with steeply decreasing components

$$\lambda_1 \gg \dots \gg \lambda_k \gg \dots \gg \lambda_m \gg \dots \gg \lambda_{|N|} > 0.$$

$\lambda$  will generate a random walk that follows  $P$  with probability  $\pi(P) \approx 1$ . The  $\lambda$ -value  $\Psi^\lambda$  therefore yields

$$\Psi_k^\lambda(v) \approx v(x_k) - v(x_{k-1}) = v(y) - v(x) < 0.$$

◇

A *symmetry* of  $\Gamma$  is a bijection  $\sigma : V \rightarrow V$  of the vertices of the transition graph  $G = (V, A)$  such that for all  $x, y \in V$  and  $i \in N$ ,

$$(S_0) \quad \sigma(xy) = \sigma(x)\sigma(y) \in A \iff xy \in A;$$

$$(S_1) \quad \sigma(A_i) = A_j \text{ for some } j \in N.$$

By  $(S_0)$ , a symmetry  $\sigma$  induces a permutation on the set  $\mathcal{P}$  of cooperation instances. So  $\sigma$  acts on the set of probability distributions  $\pi$  on  $\mathcal{P}$  (or system states of  $\Gamma$ ) via

$$\pi \mapsto \pi^\sigma \quad \text{with} \quad \pi^\sigma(P) = \pi(\sigma(P)).$$

$\sigma$  also acts on the game potentials  $v \in \mathbb{R}^V$  via

$$v \mapsto v^\sigma \in \mathbb{R}^V \quad \text{with} \quad v^\sigma(x) = v(\sigma(x))$$

in a natural way since  $\sigma(s) = s$  always holds as a consequence of  $(S_0)$ . For ease of notation, we will also write  $\pi^\sigma = \sigma(\pi)$ . By  $(S_1)$ ,  $\sigma$  induces a permutation on the set  $N$  of agents:

$$i \mapsto \sigma(i) = j \quad \text{where} \quad A_j = \sigma(A_i).$$

**Lemma 4.3** *Let  $\alpha^\pi \in \mathbb{R}^A$  be the allocation mechanism induced by the probability distribution  $\pi$  on  $\mathcal{P}$  with associated group value  $\varphi^\pi$ . Then every symmetry  $\sigma$  satisfies the identity*

$$\varphi_i^\pi(v^\sigma) = \varphi_{\sigma(i)}^{\sigma(\pi)}(v) \quad \text{for all } i \in N.$$

*Proof.* The probability that  $xy \in A$  occurs in a  $\pi$ -random walk equals the probability that  $\sigma(xy)$  occurs in a  $\sigma(\pi)$ -random walk. Hence we conclude for every  $i \in N$ ,

$$\begin{aligned} \varphi_{\sigma(i)}^{\sigma(\pi)}(v) &= \sum_{xy \in \sigma(A_i)} \alpha_{xy}^{\sigma(\pi)} [v(y) - v(x)] \\ &= \sum_{xy \in A_i} \alpha_{\sigma(xy)}^{\sigma(\pi)} [v(\sigma(y)) - v(\sigma(x))] \\ &= \sum_{xy \in A_i} \alpha_{x,y}^\pi [v^\sigma(y) - v^\sigma(x)] \\ &= \varphi_i^\pi(v^\sigma). \end{aligned}$$

◇

A *symmetry* of a game  $(\Gamma, v)$  is a symmetry  $\sigma$  of  $\Gamma$  that leaves  $v$  invariant:

$$v^\sigma(x) = v(x) \quad \text{for all } x \in V.$$

Carreras and Owen [9] have characterized the symmetries of a classical game  $v$  as those symmetries of  $\Gamma(\sigma_0)$  that are compatible with arbitrary  $\lambda$ -mechanisms. In order to generalize their result to our present model (Theorem 4.5 below), we assume (SA) as before.

**Lemma 4.4** *Assuming (SA), let  $v \in \mathbb{R}^N$  be an arbitrary game potential. Then either  $v = 0$  or there exists some  $\lambda \in \mathbb{R}^N$  with components  $\lambda_i > 0$  such that the associated  $\lambda$ -mechanism  $\alpha^\lambda$  yields a nontrivial group value, i.e.,*

$$\Psi_i^\lambda(v) \neq 0 \quad \text{for some } i \in N.$$

*Proof.* W.l.o.g. assume that  $v$  is not the null potential. So there exists some  $P = x_0x_1 \dots x_m \in \mathcal{P}$  with  $v(x_k) \neq 0$  but  $v(x_{k-1}) = 0$  for some  $k \geq 1$ . Assume furthermore  $x_{i-1}x_i \in A_i$  for all  $i = 1, \dots, m$ . As in the proof of Theorem 4.4 we choose the parameter vector  $\lambda \in \mathbb{R}^N$  so that

$$\lambda_1 \gg \dots \gg \lambda_k \gg \dots \gg \lambda_m \gg \dots \gg \lambda_{|N|} > 0.$$

Then the random walk induced by  $\lambda$  will follow  $P$  with probability  $\approx 1$  and we find

$$\Psi_k^\lambda(v) \approx v(x_k) - v(x_{k-1}) = v(x_k) \neq 0.$$

◇

**Theorem 4.5** *Assuming (SA), let  $v \in \mathbb{R}^V$  be an arbitrary potential. Then a symmetry  $\sigma$  of  $\Gamma$  is a symmetry of  $v$  if and only if*

$$\Psi_i^\lambda(v) = \Psi_{\sigma(i)}^{\sigma(\lambda)}(v)$$

*holds for all strictly positive  $\lambda \in \mathbb{R}^N$  and  $i \in N$ .*

*Proof.* If  $\sigma$  is a symmetry, i.e., if  $v = v^\sigma$ , then the condition is satisfied (Lemma 4.3). In the case  $v \neq v^\sigma$ , on the other hand, i.e., if  $w = v - v^\sigma \neq 0$ , Lemma 4.4 guarantees some  $\lambda$  and some  $i$  such that

$$0 \neq \Psi_i^\lambda(w) = \Psi_i^\lambda(v) - \Psi_{\sigma(i)}^{\sigma(\lambda)}(v)$$

violates the condition.

◇

REMARK. The statement of Theorem 4.5 may be false if assumption (SA) is dropped.

## 5 Cooperation and Interaction

We turn to the question how cooperation and interaction of agents is assessed if  $\Gamma = (N, V, A, \mathcal{A})$  is in a certain system state  $\rho$ , represented by the probability distribution  $\pi = \pi(\rho)$  on the collection  $\mathcal{P}$  of all cooperation instances. Our system theoretic model will provide a theory of *local cooperation* (Theorem 5.1). Moreover, a systematic theory for so-called interaction indices will be established.

### 5.1 Local Cooperation

Assume that the cooperation state  $x_0 \in V$  is observed. Then we are presented with the localized system

$$\Gamma(x_0) = (N, V(x_0), A(x_0), \mathcal{A}(x_0))$$

being in the system state  $\rho^0$ . Let  $\pi^0 = \pi(\rho^0)$  be the associated probability distribution on the cooperation instances of  $\Gamma(x_0)$  (see Section 2.1).

Any  $\Gamma$ -potential  $v \in \mathcal{V}$  implies *via* its marginal flow  $\partial(v)$  a  $\Gamma(x_0)$ -potential

$$v^0(x) = v(x) - v(x_0) \quad \text{for all } x \in V(x_0).$$

Denoting by  $\varphi^\pi(v|x_0) = \varphi^{\pi^0}(v^0)$  the associated random value, one might suspect that  $\varphi^\pi(v|x_0)$  results from *local cooperation values*  $\kappa_x^\pi(v)$  at the various cooperation states  $x \in V$ . It turns out that these cooperation values indeed exist and operate linearly on the potential space  $\mathcal{V}$ .

**Theorem 5.1** *For every probability distribution  $\pi$  on the cooperation instances of  $\Gamma = (N, V, A, \mathcal{A})$  and every potential  $v \in \mathcal{V}$ , there exist unique linear values  $v \mapsto \kappa_x^\pi(v) \in \mathbb{R}$  such that*

$$\varphi^\pi(v|x_0) = \sum_{x \in V(x_0)} \kappa_x^\pi(v) \quad \text{for all } x_0 \in V.$$

*Proof.* With any  $v \in \mathcal{V}$ , we associate the potential  $x \mapsto \bar{v}(x) = \varphi^\pi(v|x)$  and recall from Section 2.2 the representation

$$\bar{v}(x_0) = \sum_{x \in V} \beta_x(\bar{v}) \zeta_x(x_0) = \sum_{x \in V(x_0)} \beta_x(\bar{v})$$

in terms of simple potentials. Since  $v \mapsto \varphi^\pi(v|x)$  is linear, the operators  $v \mapsto \kappa_x^\pi(v) = \beta_x(\bar{v})$  are linear for every  $x \in V$  and have the desired properties.

◇

## 5.2 Interaction

Interaction among agents in a cooperation system  $\Gamma = (N, V, A, \mathcal{A})$  is assessed in a way similar to our analysis of local cooperation. We furthermore allow  $N$  to be partially ordered by some precedence relation  $(N, \preceq)$ , where  $i \preceq j$  indicates that  $i$  "dominates"  $j$ . A (*feasible*) *alliance* is a set  $F \subseteq N$  that respects dominance, *i.e.*, is such that for all  $i \in F$  and  $j \in N$ ,

$$i \preceq j \quad \implies \quad j \in F.$$

$\mathcal{F}$  denotes the family of all alliances. Given the cooperation state  $x_0 \in V$  and the alliance  $F \in \mathcal{F}$ , we let  $\mathcal{P}^0(F)$  be the collection of all maximal

(i.e., non-extendible) paths  $P = x_0x_1 \dots x_k$  with the property that every arc  $x_{i-1}x_i$  of  $P$  is governed by some member of  $F$ . For each such path  $P$  and potential  $v$ , recall the notation

$$\partial_P(v) = v(x_k) - v(x_0).$$

Let  $\pi$  be a probability distribution on the family  $\mathcal{P}$  of all cooperation instances.  $\pi$  induces a probability distribution  $\pi^0(F)$  on  $\mathcal{P}^0(F)$  with expected potential value

$$\partial_F^\pi(v|x_0) = \sum_{P \in \mathcal{P}^0(F)} \pi_P^0(F) \partial_P(v).$$

**Example 5.1** Let  $\Gamma(\sigma_0)$  be the full cooperation system. Then  $\partial_P(v) = v(X_0 \cup F) - v(X_0)$  holds for all  $P \in \mathcal{P}^0(F)$  and therefore

$$\partial_F^\pi(v|X_0) = v(X_0 \cup F) - v(X_0).$$

Setting  $\mathcal{F}(G) = \{F \in \mathcal{F} \mid F \subseteq G\}$ , we find:

**Theorem 5.2** Let  $(N, \preceq)$  be a dominance order,  $x_0 \in V$  a cooperation state and  $v$  a potential of  $\Gamma = (N, V, A, \mathcal{A})$ . Then there exist unique linear values  $v \mapsto \Delta_F^\pi(v|x_0)$  such that for every alliance  $G \in \mathcal{F}$ ,

$$\partial_G^\pi(v|x_0) = \sum_{F \in \mathcal{F}(G)} \Delta_F^\pi(v|x_0).$$

*Proof.* In analogy with the proof of Lemma 2.4, we define the matrix  $Z = [\zeta_{FG}] \in \{0, 1\}^{\mathcal{F} \times \mathcal{F}}$  relative to the alliance system  $\mathcal{F}$  with coefficients

$$\zeta_{FG} = 1 \iff F \subseteq G.$$

The matrix  $Z$  is invertible. Hence the claim of the Theorem may be deduced with a reasoning similar to the proof of Theorem 5.1.

◇

### 5.2.1 Interaction Values

The linear operator  $v \mapsto \Delta_F^\pi(v|x_0)$  is called the *local interaction value* of the alliance  $F \in \mathcal{F}$  in the cooperation state  $x_0 \in V$ .

**Example 5.2** Let  $\Gamma(\sigma_0)$  be the full cooperation system and assume  $N$  trivially ordered. So every subset  $F \subseteq N$  is a feasible alliance. Fix  $X_0 \subseteq N$  and two agents  $i, j \in N \setminus X_0$ . Then we find

$$\begin{aligned}\partial_i(v|X_0) &= v(X_0 \cup i) - v(X_0) \\ \partial_{ij}(v|X_0) &= v(X_0 \cup \{i, j\}) - v(X_0)\end{aligned}$$

and obtain

$$\Delta_{ij}^\pi(v|X_0) = v(X_0 \cup \{i, j\}) - v(X_0 \cup i) - v(X_0 \cup j) + v(X_0),$$

which yields Owen's [33] local interaction value. Its generalization to arbitrary subsets  $F \subseteq N$  is due to Grabisch and Roubens [21].

### 5.2.2 Interaction Indices

Given the local interaction values  $v \mapsto \Delta_F^\pi(v|x)$ , we call their weighted sum

$$v \mapsto I_F(v) = \sum_{x \in V} \beta_F(x) \Delta_F^\pi(v|x)$$

relative to a given parameter vector  $\beta_F \in \mathbb{R}^V$  an *interaction index* for the alliance  $F \in \mathcal{F}$ . Note that also interaction indices are linear values on the cooperation system and may hence be studied within the framework of the previous sections (cf. Fujimoto *et al.* [19] for the classical full cooperation model).

**Example 5.3** Let  $\Gamma(\sigma_0)$  be the full classical cooperation system. For any  $X \subseteq N$  and  $i \in N$ , we obtain the local cooperation value  $v(X \cup i) - v(X)$ . The choice

$$\beta_i(X) = \frac{1}{|N| - |X|} \binom{|N|}{|X|}^{-1}$$

yields the interaction index

$$I_i(v) = \sum_{X \subseteq N \setminus i} \frac{1}{|N|} \binom{|N| - 1}{|X|}^{-1} [v(X \cup i) - v(X)],$$

which equals the individual value of Shapley [36].

**Example 5.4** Consider  $\Gamma(\sigma_0)$ . Fix  $i, j \in N$ ,  $X \subseteq N$  and obtain the local interaction value

$$\Delta_{i,j}(v|X) = v(X \cup \{i, j\}) - v(X \cup i) - v(X \cup j) + v(X).$$

The interaction index resulting from the choice

$$\beta_{i,j}(X) = (|N| - 1)^{-1} \binom{|N| - 2}{|X|}^{-1}$$

is Owen's [33] so-called co-value:

$$I_{i,j}(v) = \sum_{X \subseteq N \setminus \{i,j\}} \beta_{i,j}(X) \Delta_{i,j}(v|X).$$

## 6 Core and Weber Sets

For our discussion of cores and Weber sets, we restrict ourselves to cooperation systems  $\Gamma = \Gamma(\sigma)$  that arise from selection structures  $(N, \sigma)$ . So each state of  $\Gamma$  corresponds to a selection  $S \subseteq N$  of agents and we define the set of *essential* members,

$$\gamma(S) = \{i \in S \mid S \setminus i \in \mathcal{S}\},$$

as consisting of those agents that could threaten to leave  $S$  in a feasible way. Given a potential  $v : \mathcal{S} \rightarrow \mathbb{R}$  on the family of selections, we define the *open core* as the non-negative polyhedron

$$\text{core}^o(v) = \{z \in \mathbb{R}_+^N \mid z(\gamma(S)) \geq v(S) \forall S \in \mathcal{S}\}, \quad (12)$$

where we employ the usual shorthand notation  $z(X) = \sum_{i \in X} z_i$ . The set  $\text{core}^o(v)$  is always non-empty. The parameter

$$v^* = \min\{z(N) \mid z \in \text{core}^o(v)\} \geq 0$$

is the minimal total allocation to the set  $N$  that would allow us to satisfy the restrictions of (12). So we define the *core* as the (non-empty) polytope

$$\text{core}^*(v) = \{z \in \text{core}^o(v) \mid z(N) = v^*\} \subseteq \mathbb{R}_+^N.$$

## 6.1 Weber Sets

Let  $B$  be a  $\sigma$ -basis (and hence a sink in the transition graph) and consider the source-sink path  $P \in \mathcal{P}_B$ :

$$P : \emptyset = S_0 \subset S_1 \subset \dots \subset S_{n-1} \subset S_n = B.$$

Fix a potential  $v$  and assume that the indices have been chosen so that  $i \in \gamma(S_i)$  holds for  $i = 1, \dots, n$ . Define the upper triangular  $(n \times n)$ -incidence matrix  $M = M(P)$  with coefficients

$$m_{ij} = \begin{cases} 1 & \text{if } i \in \gamma(S_j) \\ 0 & \text{otherwise.} \end{cases}$$

$M$  is of full rank  $n$  and invertible. So there is a unique vector  $h^P$  such that

$$\sum_{j \in \gamma(S_i)} h_j^P = v(S_i) \quad (i = 1, \dots, n).$$

We extend  $h^P$  to all of  $N$  via  $h_j^P = 0$  for  $j \in N \setminus S_n$  and call  $h^P$  a *marginal vector*. We define the *Weber set* relative to the basis  $B$  as the convex of all marginal vectors:

$$\mathcal{W}_B(v) = \text{conv}\{h^P \in \mathbb{R}^N \mid P \in \mathcal{P}_B\}.$$

## 6.2 Greedy Paths and Potentials

Let now  $c \in \mathbb{R}^N$  be an arbitrary parameter vector. We construct a *greedy path*  $P$  and a *greedy potential*  $y$  recursively from a basis of  $(N, \sigma)$  towards the empty set as follows.

- ( $G_0$ ) Choose some basis  $B \in \mathcal{S}$  and initialize:  
 $P \leftarrow B$ ;  $y(S) \leftarrow 0$  for all  $S \in \mathcal{S}$ ;  $X \leftarrow B$ ;
- ( $G_1$ ) If  $X = \emptyset$ , output  $(P, y)$  and stop.
- ( $G_2$ ) If  $X \neq \emptyset$ , choose  $i \in \gamma(X)$  of minimal weight  $c_i$ ;  
 Update:  $y(X) \leftarrow c_i$ ;  $c_j \leftarrow [c_j - c_i]$  for all  $j \in \gamma(X)$ ;  $P \leftarrow XP$ ;
- ( $G_3$ ) Update  $X \leftarrow [X \setminus i]$  and goto ( $G_1$ );

**Lemma 6.1** *Let  $P = \emptyset S_1 \dots S_n$  be the path with potential  $y \in \mathbb{R}^V$  constructed by the greedy algorithm relative to the non-negative parameter vector  $c \in \mathbb{R}_+^N$ . Assume furthermore that the agents are labeled so that*

$i \in \gamma(S_i)$  holds. Then  $y(S) \geq 0$  holds for all  $S \in \mathcal{S}$  and, in particular,  $y(S) = 0$  for all  $S \notin P$ . Moreover,  $y$  satisfies the linear constraints

$$\sum_{\gamma(S) \ni i} y(S) \begin{cases} = c_i & \text{for all } i = 1, \dots, n \\ \leq c_i & \text{otherwise.} \end{cases}$$

*Proof.* The greedy algorithm chooses agent  $n \in \gamma(S_n)$  of minimal weight. This choice implies  $y(S_n) = c_n \geq 0$  and  $c_j - c_n \geq 0$  for all  $j \in \gamma(S_n)$ , i.e., the updated  $c$ -values remain non-negative for all the agents. Hence also  $y(S_{n-1}) \geq 0$  is guaranteed etc., which yields  $y(S) \geq 0$  for all selections  $S \in \mathcal{S}$ .

The weight  $c_i$  of agent  $i$  changes in an update only if  $i \in \gamma(X)$  holds for the current  $X$ . Moreover, the weight is never increased. Hence we find equality to hold for  $1 \leq i \leq n$ :

$$c_i = \sum \{y(S_j) \mid j \geq i, i \in \gamma(S_j)\} = \sum_{\gamma(S) \ni i} y(S).$$

Inequality follows the same way. ◇

**Lemma 6.2** *Let  $(P, y)$  be the output of the greedy algorithm relative to the weight vector  $c \in \mathbb{R}_+^N$ . Then the associated marginal vector  $h = h^P$  satisfies*

$$\sum_{j \in N} c_j h_j = \sum_{S \in P} v(S) y(S) = \sum_{S \in \mathcal{S}} v(S) y(S).$$

*Proof.* Let  $M = M(P)$  be the incidence matrix of  $P$ . Then we have  $h^T M = v^T$  (restricted to the selections  $S \in P$ ) and  $M y = c$  (Lemma 6.1) and therefore

$$c^T h = y^T M^T h = y^T v. \quad \diamond$$

**Theorem 6.1** *Let  $B$  be a basis in  $\Gamma(\sigma)$  and  $v \geq 0$  an arbitrary potential of  $\Gamma(\sigma)$ . Then the following holds.*

- (a) *If  $h^P(N) < v^*$  holds for all  $P \in \mathcal{P}_B$ , then  $\text{core}^*(v) \cap \mathcal{W}_B(v) = \emptyset$ .*
- (b) *If  $h^P(N) \geq v^*$  holds for all  $P \in \mathcal{P}_B$ , then  $\text{core}^*(v) \subseteq \mathcal{W}_B(v)$ .*

*Proof.* Assume  $z \in \text{core}^*(v) \cap \mathcal{W}_B(v)$  exists. Since  $z$  is a convex combination of marginal vectors, there must be at least one marginal vector  $h$  with  $h(N) \geq z(N) = z^*$ , which proves (a).

Suppose now that claim (b) of the Theorem is false and a vector  $z \in \text{core}^*(v) \setminus \mathcal{W}_B(v)$  exists. Since  $\mathcal{W}_B(v)$  is a closed convex set, there is a hyperplane separating  $z$  from  $\mathcal{W}_B(v)$ , *i.e.*, there is a parameter vector  $c \in \mathbb{R}^N$  so that

$$\sum_{j \in N} c_j z_j < \sum_{j \in N} c_j h_j^P \quad \text{for all paths } P \in \mathcal{P}_B.$$

Because of  $0 \leq z(N) = v^* \leq h^P(N)$ , we may assume  $c \geq 0$  w.l.o.g. (Otherwise, we add a large enough constant  $C > 0$  to each component of  $c$  without affecting the inequality). Run the greedy algorithm with respect to  $c$  and obtain the greedy path  $P \in \mathcal{P}_B$ , greedy potential  $y$  and the marginal vector  $h^P$ .

Because of  $c, y, z \geq 0$ , we deduce from Lemma 6.1 the inequalities

$$\begin{aligned} \sum_{j \in N} c_j z_j &\geq \sum_{j \in N} \left( \sum_{\gamma(S) \ni j} y(S) \right) z_j = \sum_{S \in \mathcal{S}} \sum_{j \in \gamma(S)} y(S) z_j \\ &\geq \sum_{S \in \mathcal{S}} y(S) v(S) = \sum_{j \in N} c_j h_j^P, \end{aligned}$$

where the last equality follows from Lemma 6.2. This contradiction establishes the Theorem.  $\diamond$

**Corollary 6.1** *Let  $B$  be a basis and  $v$  a potential of  $\Gamma(\sigma)$ . Then the following statements are equivalent:*

- (i)  $\text{core}^*(v) = \mathcal{W}_B(v)$ .
- (ii)  $h^P \in \text{core}^*(v)$  for all  $P \in \mathcal{P}_B$ .

*Proof.* (ii) obviously follows from (i). Conversely, if (ii) holds, then  $h^P(N) = v^*$  is true. So statement (b) of Theorem 6.1 yields

$$\text{core}^*(v) \subseteq \mathcal{W}_B(v) \subseteq \text{core}^*(v)$$

since  $\text{core}^*(v)$  is a convex set.  $\diamond$

### 6.2.1 The Classical Core

Consider the full cooperation system  $\Gamma(\sigma_0)$  with  $\sigma_0(X) = N \setminus X$  and hence  $\gamma(X) = X$  for all  $X \subseteq N$ . The classical core model attempts to distribute the value  $v(N)$  relative to a game potential  $v$  among the members of  $N$  and therefore investigates the non-negative polytope

$$\text{core}(v) = \{z \in \mathbb{R}_+^N \mid z(N) = v(N), z(X) \geq v(X) \forall X \subseteq N\}. \quad (13)$$

In view of  $v^* \geq v(N)$ , it is clear that the polytope  $\text{core}(v)$  can only be non-empty if  $v^* = v(N)$  is guaranteed. So we deduce from Theorem 6.1:

**Corollary 6.2 (Weber [38])** *For the full cooperation system  $\Gamma(\sigma_0)$ , one has*

$$\text{core}(v) \subseteq \mathcal{W}_N(v) \quad \text{for every game potential } v.$$

*Proof.* If  $v(N) < v^*$ , then  $\text{core}(v) = \emptyset \subseteq \mathcal{W}_N(v)$  is trivially true.  $\diamond$

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