A System-theoretic Model for Cooperation and Allocation Mechanisms

Ulrich Faigle and Jan Voss

Department of Mathematics
University of Cologne

Cologne Twente Workshop, 2009

Outline

Outline

1 Introduction

2 The Model - Of Systems and States

3 Allocations and Values

4 Monotone Potentials & Symmetries - or: How to use λ-Values?

The Classical Model

Assumption of classical cooperative game theory:

- Arbitrary subsets of agents can join to form feasible coalitions and create *values* in a given economic context.

So a cooperative game is a pair (N, v) consisting of

- a finite set N of *agents* and
- a valuation $v : 2^N \to \mathbb{R}$ of the *coalitions* with $v(\emptyset) = 0$.

The Main Question of Cooperative Game Theory

Main question:
How should a commonly generated value be distributed in a *fair* way among the agents?(i.e.: What is a *fair* payoff vector in $\mathbb{R}^N$ for a game?)
Classical solution concepts for this question are for example:

- The *core* of a game.
- The Shapley value (SHAPLEY, 1953).
- Random order values (WEBER 1988).

The Classical Model

The classical model does not suit for many practical situations. For example situations in which

- not all subsets of N are possible coalitions,
- a hierachy or a ranking of the agents is important or
- dynamic processes are involved.

Previous Generalizations & Extensions

Examples

- KALAI and SAMET, 1987.
- FAIGLE and KERN, 1992.DERKS and GILLES, 1995.
- BILBAO *et al.*, 1999-2004.

All these models give generalizations of the classical model, the Shapley value and possibly of the core.

But There is One Problem...

Problem:

Each model deals with its own special setting in which the classical model does not suffice. But there is no model, which covers all these generalizations.

Outline

Cooperation Systems

A *cooperation system* is a quadruple $\Gamma = (N, V, A, \mathcal{A})$, where

- N is a finite set of *agents*,
- V is a finite set of *states of cooperation*,
- A a finite set of *feasible transitions* $x \to y$ between states in V and
- $\mathcal{A} := \{A_i | i \in N\}$ $(A_i \subseteq A)$ is a partition of A.

Transition Graphs

A cooperation system Γ is (in a natural way) based on a directed graph:

- Identify transitions $(x \to y)$ with ordered pairs $xy \in V \times V$.
- Obtain the (directed) transition Graph $G = (V, A)$ of Γ.

Notation: For $x \in V$ set
$x^{-} := \{u \in V | ux \in A\}$ and $x^{+} := \{u \in V | xu \in A\}$.

Two Simplifying Assumptions

We assume throughout

(Γ_0) There is one unique initial state $s \in V$ with $s^- = \emptyset$.

(Γ_1) $G = (V, A)$ is acyclic.

Easy consequences:

- Every $x \in V$ can be reached via a directed path starting in s.
- Every path extends to a path that starts in s and ends in a sink t (i.e.: $t^+ = \emptyset$).

Denote by $\mathcal{P}$ the set of all source sink paths in G.

Cooperative Games

A *cooperative game* is a pair (Γ, v), where

- Γ is a cooperation system and
- $v \in \mathbb{R}^V$ is a valuation of the states of cooperation.
- v is zero-normalized (i.e.: $v(s) = 0$).

Denote by $\mathcal{V}(= \mathcal{V}(\Gamma))$ the vector space of all game potentials on Γ.

The Classical Model is a Cooperation system

The classical Model is a cooperation system with

- $V := 2^N$ and
- $A_i := \{(S, S \cup \{i\}) | S \in 2^N, i \notin S\}$.

The source is $s = \emptyset$ and the only sink is $t = N$.

Remark: The previous mentioned models are also cooperation structures.

Outline

1 Introduction

2 The Model - Of Systems and States

3 Allocations and Values

4 Monotone Potentials & Symmetries - or: How to use λ-Values?

Equivalent Potentials & The Marginal Operator

Two more definitions:

- $\partial : \mathbb{R}^V \to \mathbb{R}^A$, where

  $$\partial_{xy}(v) = v(y) - v(x) \text{ for all } xy \in A,$$

  is called the *marginal operator*.
- We call potentials $v, w \in \mathbb{R}^V$ *equivalent*, if $v - w$ is constant.

Lemma

v and w are equivalent, if and only if $\partial(v) = \partial(w)$.

Allocations I

An allocation mechanism is a computational sheme for allocating payoffs to the agents $i \in N$ with the following properties:

- The game $v \equiv 0$ should yield zero payoff.
- The allocation should be linear in v.
- The allocation to agent i should only depend on A_i.

Allocations II

These assumptions imply:

- Equivalent games produces the same allocation (i.e. the allocation rel. to $v \in \mathcal{V}$ depends only on $\partial(v)$).

Since $\mathcal{V} \simeq \partial(\mathcal{V})$:

- A linear allocation mechanism is described by a vector $\alpha \in \mathbb{R}^A$ that determines the individual values for $i \in N$:

$$\phi_i^\alpha(v) = \sum_{xy \in A_i} \alpha_{xy} \partial_{xy}(v).$$

Group Values & Efficiency

We call the linear function

$$v \mapsto \phi^{\alpha}(v) := \alpha^T \partial(v) (= \sum_{i \in N} \phi_i^{\alpha}(v))$$

the *group value* associated with the allocation mechanism α.
α is called efficient if there are $\mu_t \in \mathbb{R}$ such that

(E) $\sum_{t \in T} \mu_t = 1$ and $\phi^{\alpha}(v) = \sum_{t \in T} \mu_t v(t)$ for all $v \in \mathcal{V}$.

Group Values & Efficiency

Example: In the classical model a group value is called efficient, if $\phi(v) = v(N)$. Since $t = N$ is the only sink, this notion agrees with our definition.

Theorem

The allocation mechanism $\alpha \in \mathbb{R}^A$ is efficient if and only if α is an s-flow with the property $\alpha^+(s) = 1$.

(α is a s-flow, if $\sum_{u \in x^-} f_{ux} = \sum_{u \in x^+} f_{xu}$ for all inner vertices $x \in V$. $\alpha^+(s) := \sum_{x \in s^+} \alpha_{sx}$.)

α-Random Walks

Let $\alpha \in \mathbb{R}^A$ be an efficient allocation mechanism with $\alpha_{xy} \geq 0$ for all $xy \in A$. Perform a α-random walk by

(R_0) The walk starts in the initial state s.

(R_1) The walk moves from x to y over the arc $xy \in A$ with probability $\pi_{xy} = \alpha_{xy}/\alpha^+(x)$.

(R_2) The walk stops, if any final state is reached.

A Stochastic Interpretation of Values

Lemma

Let $\alpha \in \mathbb{R}^A$ be non-negative and efficient. Then each component α_{xy} of α equals the probability that the associated α-random walk passes through $xy \in A$.

This yields a stochastic interpretaion of non-negative efficient linear values:

$$\phi_i^\alpha(v) = \sum_{xy \in A_i} \alpha_{xy} \partial_{xy}(v)$$

is exactly the expected payoff total of agent $i \in N$.

Random Values

Lemma

Let π *be a probability distribution on* $\mathcal{P}$*. Set* $\alpha_{xy} := \sum_{P \ni xy} \pi_P$*.*
Then α *is an efficient and non-negative allocation mechanism.*

Together with the previous lemma we get:

Theorem

Probability distributions on $\mathcal{P}$ *correspond to non-negative efficient allocation mechanisms.*

Shapley Allocations

Let $(\mu_t)_{t\in T}$ be a probability distribution on the set T of final states and consider the functional

$$v \mapsto \phi(v) = \sum_{t\in T} \mu_t v(t)$$

μ induces a probability distribution on $\mathcal{P}$ via

$$\pi_P^S(\mu) := \frac{\mu_t}{|\mathcal{P}_t|} \text{ if } P \in \mathcal{P}_t.$$

We call the associated mechanism, α^μ, the *Shapley allocation mechanism* relative to μ. This generalizes the Shapley values thought of in many previous generalizations.

Characterization of Shapley Allocations

Theorem

α^μ is the unique non-negative efficient allocation mechanism α of maximal entropy with $\phi^\alpha = \phi$.

(The entropy of α is the entropy of the induced probability distribution on $\mathcal{P}$:

$$H(\alpha) := H(\pi^\alpha) := -\sum_{P\in\mathcal{P}} \pi_P^\alpha log_2(\pi_P^\alpha))$$

λ-Mechanisms

Assume that the agents are weighted by a positive vector $\lambda \in \mathbb{R}^N$. Set

$$\alpha_{xy}^{\lambda} := \frac{\lambda_i}{\lambda^+(x)} \text{ for all } xy \in A_i.$$

Denote by Ψ^λ the associated group value.

Remark: These λ-values are precisely the weighted Shapley values in the model of KALAI and SAMET.

Outline

1. Introduction
2. The Model - Of Systems and States
3. Allocations and Values
4. Monotone Potentials & Symmetries - or: How to use λ-Values?

Single Action & Monotone Games

$\Gamma = (N, V, A, \mathcal{A})$ has the *single action property* if

(SA) $|P \cap A_i| \leq 1$ for all $P \in \mathcal{P}$ and $i \in N$.

We call a potential v of Γ monotone if

$$xy \in A \Rightarrow v(x) \leq v(y)$$

holds for all $x, y \in V$.

Theorem

Assume that (SA) holds. Then a potential v on Γ is monotone if and only if $\Psi_i^\lambda(v) \geq 0$ for all λ-values Ψ^λ and all $i \in N$.

Symmetries

A *symmetry* of Γ is a bijection $\sigma : V \to V$ such that for all $x, y \in V$ and $i \in N$,

(S_1) $xy \in A \Leftrightarrow \sigma(x)\sigma(y) \in A$

(S_2) $\sigma(A_i) = A_j$ for some $j \in N$.

A symmetry σ acts in a natural way on

- $\mathcal{V}$ via $v^\sigma(x) := v(\sigma(x))$
- N via $\sigma(i) := j$ if $\sigma(A_i) = \sigma(A_j)$
- on $\lambda \in \mathbb{R}^N$ by permuting its components.

σ is called an automorphism of the potential v if $v^\sigma = v$.

A Characterization of Automorphisms

Theorem

Assume (SA) and let $v \in \mathbb{R}^V$. Then the symmetry σ of Γ is an automorphism of v if and only if

$$\Psi_i^{\lambda}(v) = \Psi_{\sigma(i)}^{\sigma(\lambda)}(v)$$

holds for all $\lambda \in \mathbb{R}_+^N$ and $i \in N$.

Remarks:

- In the classical case this theorem was discovered by CARRERAS and OWEN(1997).
- Sadly the statement may be false if (SA) is dropped. ☹

Anything else?

Things that are not in this talk - but in the paper:

- Generalizations of the Core & Weber sets. (Also a generalization of the famous classical theorem of WEBER: core$\subseteq$ Weber set)
- Local cooperation, interaction and Interaction indices.

For further reading: http://www.zaik.de/∼voss/

Thank you
for your attention.

$$\ddot{\smile}$$