AN ALGEBRAIC FRAMEWORK FOR THE GREEDY ALGORITHM WITH APPLICATIONS TO THE CORE AND WEBER SET OF COOPERATIVE GAMES

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Abstract. An algebraic model generalizing submodular polytopes is presented, where modular functions on partially ordered sets take over the role of vectors in $\mathbb{R}^n$. This model unifies various generalizations of combinatorial models in which the greedy algorithm and the Monge algorithm are successful and generalizations of the notions of core and Weber set in cooperative game theory.

As a further application, we show that an earlier model of ours as well as the algorithmic model of Queyranne, Spieksma and Tardella for the Monge algorithm can be treated within the framework of usual matroid theory (on unordered ground-sets), which permits also the efficient algorithmic solution of the intersection problem within this model.

1. Introduction

Matroids are characterized by the fact that they admit a simple greedy algorithm for the optimization of linear weight functions. Analyzing the matroid greedy algorithm in the setting of linear programming, Edmonds [1970] showed that it can be generalized to combinatorial systems presented by submodular set functions (see also Fujishige [1991]). An even more general framework (lacking the full algorithmic counterpart, however) for the analysis of combinatorial systems are the lattice polyhedra under submodular constraints of Hoffman [1982]. On the other hand, also the Monge algorithm can be viewed as some kind of greedy algorithm involving some kind of submodularity although it is not a matroid greedy algorithm (*cf.* Hoffman [1985]).

A more general model for submodular systems has been introduced by Queyranne *et al.* [1998] allowing a common framework for both the matroid greedy algorithm and the Monge algorithm. These authors show in particular that their model includes the model of Faigle and Kern [1996] for optimizing linear functions under submodular constraints relative to antichains of rooted forests. The case of unordered ground-sets (*i.e.*, trivial rooted forests) corresponds to the usual model of submodular systems.

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Motivated by the seemingly different problem of developing economic models for the "fair" allocation of costs or profits, set functions offer the basic model of games in the sense of cooperative game theory. Allocations to individual players can be viewed as vectors satisfying certain constraints. Important classes of cooperative games are described by submodular set functions. The core of such a submodular game then is exactly a submodular system in the sense above. A related solution concept is the Weber set of a game. It turns out that the Weber set is always non-empty, contains the core and coincides with the core if the game is submodular (*cf.* Weber [1988]). There have been various investigations into models of cooperative games that also reflect hierarchies among the players (see, *e.g.* Faigle and Kern [1992] or Derks and Gilles [1995]). Models based on an even more general combinatorial sub-structure of closure spaces have been proposed by Bilbao [1998] and Jiménez [1998].

Modeling an "information theory of value", Danilov *et al.* [1999] are lead to the consideration of modular functions on lattices instead of allocation vectors. They are able to present an equilibrium theory in the case of modular functions on distributive lattices under submodular constraints by extending the appropriate concepts of submodular systems suitably.

The purpose of the present paper is to propose a general common model of modular functions on posets equipped with a binary operation as a general framework for the models above. The latter can be recovered via appropriate set-theoretic representations of the general model. Our approach not only unifies various algorithmic generalizations of the basic matroid model but also attempts to exhibit more clearly the structural connections between basic problems in cooperative game theory and combinatorial optimization. In this way, we are not only able to derive generalizations of various known results. Our approach by modular functions also shows, for example, that the model of Queyranne *et al.* [1998] is structurally equivalent with the usual model of submodular functions on unordered ground-sets. An important consequence is the (new) result that also the intersection problem relative to two submodular functions in the model of Queyranne *et al.* is solvable by an efficient algorithm.

We must note, however, that our present model is still unable to provide a full algorithmic theory for submodular lattice polyhedra. Another interesting open problem concerns the primal-dual greedy algorithm of Frank [1998] that seems to bear some formal resemblance with our present greedy algorithm in Section 6. . Yet, neither of the two appears to imply the other. Does there exist a common algorithmic generalization?

The paper is organized as follows. In Section 2, we review the duality theory of linear programming in our general context. Submodular functions on algebraic posets are introduced in Section 3. Set-theoretic representations of these structures are discussed in Section 4. In Section 5, we derive general theorems on the core and Weber set in our framework. Algorithmic aspects are deferred to Section 6.

2. Abstract Cone Duality

In this section, we will discuss a natural abstraction of the model of linear programming.

Let $\mathcal{L}$ be a finite set. We consider the vector space $\mathcal{V} = \mathbb{R}^{\mathcal{L}}$ of all real-valued functions

$$f : \mathcal{L} \to \mathbb{R} \,.$$

By $\mathcal{V}_+ = \mathbb{R}^{\mathcal{L}}_+$, we denote the convex cone of all non-negative functions in $\mathcal{V}$.

Let $f \in \mathcal{V}$ be a fixed element of $\mathcal{V}$, $\mathcal{M} \subseteq \mathcal{V}$ a linear subspace and

$$\gamma : \mathcal{M} \to \mathbb{R}$$

a linear map on $\mathcal{M}$. Then the triple $(f, \mathcal{M}, \gamma)$ gives rise to the optimization problem

$$\text{(P)} \qquad \begin{array}{rrcl} \max & \gamma(m) & & \\ s.t. & f - m & \in & \mathcal{V}_+ \\ & m & \in & \mathcal{M} \,. \end{array}$$

Denote by $\mathcal{V}^*$ the dual vector space of all linear functionals $\gamma : \mathcal{V} \to \mathbb{R}$. Moreover, let the *dual cone* $\mathcal{V}^*_+$ consist of all linear maps $\rho \in V^*$ such that

$$\rho(h) \;\geq\; 0 \quad \text{holds for all } h \in V_+ \,.$$

Then we can formulate the *dual* optimization problem relative to (P) as

$$\text{(D)} \qquad \begin{array}{rrcll} \min & \rho(f) & & & \\ s.t. & \rho & \in & \mathcal{V}^*_+ & \\ & \rho(m) & = & \gamma(m) & \text{for all } m \in \mathcal{M}. \end{array}$$

If $m \in \mathcal{M}$ is such that $h = f - m \in \mathcal{V}_+$ and if $\rho \in \mathcal{V}^*_+$, then we have, by definition, $\rho(h) \geq 0$, *i.e.*, $\rho(f) \geq \rho(m)$. This observation yields the well-known

Lemma 2.1. *("Weak Duality"): If m is feasible for (P) and ρ is feasible for (D), then*

$$\rho(f) \;\geq\; \gamma(m) \,.$$

$\diamond$

Optimality for both (P) and (D) thus is given if we can construct feasible m and ρ so that $\gamma(m) = \rho(f)$ holds. We will derive a sufficient condition.

The *standard basis* of $\mathbb{R}^{\mathcal{L}}$ is provided by the indicator functions χ_L, $L \in \mathcal{L}$, where for all $A \in \mathcal{L}$,

$$\chi_L(A) \;=\; \begin{cases} 1 & \text{if } A = L \\ 0 & \text{otherwise.} \end{cases}$$

Thus every $f \in \mathbb{R}^{\mathcal{L}}$ can be uniquely represented as

$$f \;=\; \sum_{L\in\mathcal{L}} f(L)\chi_L \;,$$

and every $\rho \in \mathcal{V}^*$ is uniquely determined by the coefficients

$$\rho_L \;:=\; \rho(\chi_L) \quad \text{for all } L \in \mathcal{L} \;.$$

Defining the *support* of $\rho \in \mathcal{V}^*$ as

$$\operatorname{supp}(\rho) \;:=\; \{L \in \mathcal{L} \mid \rho_L \neq 0\} \;,$$

we observe

$$\rho(f) = \sum_{L\in\mathcal{L}} f(L)\rho_L = \sum_{L\in supp(\rho)} f(L)\rho_L \;.$$

This yields another well-known result.

Lemma 2.2. *("Complementary Slackness"): Let $\rho \in \mathcal{V}^*$ be such that $\rho(m) = \gamma(m)$ for all $m \in \mathcal{M}$. Let furthermore $m_f \in \mathcal{M}$ satisfy*

$$m_f(L) = f(L) \quad \textit{for all } L \in supp(\rho).$$

Then $\gamma(m_f) \;=\; \rho(f)$.

Proof.

$$\rho(f) = \sum_{L\in supp(\rho)} f(L)\rho_L = \sum_{L\in supp(\rho)} m_f(L)\rho_L = \rho(m_f) = \gamma(m_f) \;.$$

$\diamond$

In view of Lemma 2.2, our strategy for solving (P) optimally will be to exhibit feasible solutions ρ for (D) so that we can find feasible solutions m_f for (P) with

$$m_f(L) = f(L) \quad \text{for all } L \in \operatorname{supp}(\rho) \;.$$

We remark that our optimization problems (P) and (D) can be viewed as a primal-dual pair of linear programs in the usual sense. To see this, choose a basis $\{m_1, \ldots, m_n\}$ for the subspace $\mathcal{M}$ and let $M = (m_i(L))$ be the $(|\mathcal{L}| \times n)$-matrix arising from the representations

$$m_i \;=\; \sum_{L\in\mathcal{L}} m_i(L)\chi_L \;.$$

If $c \in \mathbb{R}^n$ is the vector with coefficients

$$c_i \;=\; \gamma(m_i) \;,$$

then (P) amounts to the linear program

$$\begin{array}{lrcl} \max & c^T x & & \\ s.t. & Mx & \leq & f \\ & x & \in & \mathbb{R}^n \end{array}$$

Dually, the condition $\rho \in \mathcal{V}_+^*$ means $\rho_L = \rho(\chi_L) \geq 0$ for all $L \in \mathcal{L}$. Identifying ρ with the vector $(\rho_L)_{L\in\mathcal{L}}$, and f with the vector $(f(L))_{L\in\mathcal{L}}$, (D) is seen to be equivalent with the dual linear program

$$\begin{array}{llcl} \min & \rho^T f & & \\ s.t. & \rho & \geq & 0 \\ & \rho^T M & = & c^T \end{array} .$$

3. Algebraic Posets and Submodular Functions

Let $(\mathcal{L}, \leq)$ be a (finite) partially ordered set ("poset"). We assume that there is a (partially defined) binary operation "$\wedge$" on $\mathcal{L}$ such that $A \wedge B \in \mathcal{L}$ whenever $A \wedge B$ is defined at all for the elements $A, B \in \mathcal{L}$. For any given $B, C \in \mathcal{L}$, we say that C is an *upper neighbor* of B if $B < C$ holds and there is no $Z \in \mathcal{L}$ with $B < Z < C$.

We say that three elements $A, B, C \in \mathcal{L}$ form an *algebraic triple* in $(\mathcal{L}, \leq)$ if A and B are incomparable and C is an upper neighbor of B satisfying $A < C$. $(\mathcal{L}, \leq, \wedge)$ is defined to be an *algebraic* poset if for each algebraic triple (A, B, C), $A \wedge B$ is defined and satisfies $A \wedge B \leq B$.

Convention: When writing $A \wedge B$ for two given elements $A, B \in \mathcal{L}$, we in particular tacitly imply that $A \wedge B$ is defined.

We denote the algebraic poset by "$(\mathcal{L}, \leq, \wedge)$" in order to emphasize the structure, but we will often refer to it simply as "$\mathcal{L}$" if the structural context is clear.

Assumption: $\mathcal{L}$ has a (unique) maximal element $\mathbf{1} \in \mathcal{L}$ (satisfying $A \leq \mathbf{1}$ for all $A \in \mathcal{L}$).

Example 3.1. *An ordered set $(P, \leq)$ is a* semilattice *if for every two elements $a, b \in P$, there exists a unique maximal element $a \wedge b \leq a, b$. If the semilattice P has a unique maximal element, then P is a* lattice *(cf Birkhoff [1967] for more details). Clearly, every lattice is an algebraic poset.*

Consider the function $f : \mathcal{L} \to \mathbb{R}$ on the algebraic poset $\mathcal{L}$.

We say that f is *submodular* if for all algebraic triples (A, B, C) of $\mathcal{L}$,

$$f(A) - f(A \wedge B) \;\geq\; f(C) - f(B) .$$

f is *supermodular* if the function $g = -f$ is submodular. f is finally *modular* if f is both submodular and supermodular.

With these definitions, we arrive at a fundamental observation.

Theorem 3.1. *Let $\mathcal{C} = \{C_1 < \ldots < C_i < \ldots < C_n\}$ be a maximal chain of the algebraic poset $\mathcal{L}$ and $f \in \mathbb{R}^{\mathcal{L}}$ a submodular function satisfying $f(C_i) = 0$ for all $i = 1, \ldots, n$. Then*

$$f(A) \;\geq\; 0 \;\; \textit{for all } A \in \mathcal{L} .$$

Proof. Set $\mathcal{L}_i := \{L \in \mathcal{L} \mid L \leq C_i\}$, $i = 1, \ldots, n$. Thus $\mathcal{L}_1 = \{C_1\}$ and $\mathcal{L}_n = \mathcal{L}$.

By induction on i, assume $f(L) \geq 0$ holds for all $L \in \mathcal{L}_{i-1}$ and consider $A \in \mathcal{L}_i \setminus \mathcal{L}_{i-1}$. Then (A, C_{i-1}, C_i) is an algebraic triple. Hence the submodularity of f implies

$$f(A) \;\geq\; f(C_i) - f(C_{i-1}) + f(A \wedge C_{i-1}) \;\geq\; 0 \,.$$

$\diamond$

Corollary 3.1. *Let $\mathcal{C}$ be a maximal chain of the algebraic poset $\mathcal{L}$, and let $f, g \in \mathbb{R}^{\mathcal{L}}$ be functions such that $f(C) = g(C)$ for all $C \in \mathcal{C}$. Then*

(a) *If f is submodular and g supermodular, then $f \geq g$.*
(b) *If both f and g are modular, then $f = g$.*

Proof. Apply Theorem 3.1 to the function $h = f - g$.

$\diamond$

There exists a certain converse to Corollary 3.1.

Theorem 3.2. *Let $(\mathcal{L}, \leq, \wedge)$ be an algebraic poset, and let $f \in \mathbb{R}^{\mathcal{L}}$ be such that for every maximal chain $\mathcal{C} \in \mathcal{L}$, there exists some modular $m \in \mathbb{R}^{\mathcal{L}}$ with $m \leq f$ and $m(C) = f(C)$ for all $C \in \mathcal{C}$. Then f is submodular.*

Proof. Let (A, B, C) be an algebraic triple in $\mathcal{L}$. Choose a maximal chain $\mathcal{C}$ containing C, B, and $A \wedge B$. By assumption, there exists a modular function $m \in \mathbb{R}^{\mathcal{L}}$ such that $m \leq f$ and $m(C') = f(C')$ for all $C' \in \mathcal{C}$. Then

$$\begin{aligned} f(A) &\geq m(A) \\ &= m(C) - m(B) + m(A \wedge B) \\ &= f(C) - f(B) + f(A \wedge B) \qquad . \end{aligned}$$

$\diamond$

We remark that our definition of "submodularity" differs from the usual definition, which refers to *two* binary operations $\wedge, \vee : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and calls a function $f : \mathcal{L} \to \mathbb{R}$ "submodular" if for all $A, B \in \mathcal{L}$,

$$f(A) + f(B) \;\geq\; f(A \wedge B) + f(A \vee B)$$

holds. Let us call such a function f *globally submodular* (with respect to $(\mathcal{L}, \wedge, \vee)$). In order to explore the relation between global submodularity and submodularity as introduced here, we call the binary operation

$$\vee : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

upper algebraic if for each algebraic triple (A, B, C) in $\mathcal{L}$,

$$A \vee B \;=\; C \,.$$

Proposition 3.1. *Let $(\mathcal{L}, \leq, \wedge, \vee)$ be such that $(\mathcal{L}, \leq, \wedge)$ is an algebraic poset and $\vee : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ upper algebraic. Then every globally submodular function $f \in \mathbb{R}^{\mathcal{L}}$ is submodular.*

Proof. Let (A, B, C) be an algebraic triple in $\mathcal{L}$. If f is globally submodular, we have

$$f(C) = f(A \vee B) \leq f(A) + f(B) - f(A \wedge B) .$$

$\diamond$

Note, on the other hand, that a submodular function f need not be globally submodular.

Example 3.2. *Let E be a finite set and $\mathcal{L}$ a family of subsets of E such that $E \in \mathcal{L}$ and $A \cap B \in \mathcal{L}$ for all $A, B \in \mathcal{L}$. Order $\mathcal{L}$ by set-theoretic containment and define for all $A, B \in \mathcal{L}$,*

$$A \wedge B := A \cap B .$$

Then $\mathcal{L}$ is an algebraic poset. In fact, $\mathcal{L}$ is a lattice (cf. Example 3.1).

Define for every $A, B \in \mathcal{L}$,

$$A \vee B := \bigcap \{L \in \mathcal{L} \mid A \cup B \subseteq L\} .$$

For every $e \in E$ we consider the indicator function $\chi_e \in \mathbb{R}^{\mathcal{L}}$, *given by*

$$\chi_e(L) \;=\; \begin{cases} 1 & \textit{if } e \in L \\ 0 & \textit{if } e \notin L . \end{cases}$$

Then it is not difficult to see that every indicator function is (globally) supermodular. Moreover, every indicator function is modular if and only if $\mathcal{L}$ is locally distributive *in the sense that for every algebraic triple (A, B, C) of $\mathcal{L}$,*

$$C \;=\; A \cup B .$$

However, every indicator function is globally modular relative to $(\mathcal{L}, \wedge, \vee)$ if and only if $\mathcal{L}$ is closed under set-theoretic union (i.e., $A \vee B = A \cup B$ for all $A, B \in \mathcal{L}$).

$\diamond$

We remark that intersection-closed set systems $\mathcal{L}$ containing the groundset are also known as *closure systems.* Locally distributive closure systems such that $|C| = |B| + 1$ if C is an upper neighbor of B are known as "convex geometries". Our concept of "submodularity" generalizes the concept of "quasi-submodularity" introduced by Jiménez [1998] for convex geometries. (For more information on the fundamental role of convex geometries, or, equivalently, "antimatroids" in the theory of finite geometries and combinatorial optimization resp., we refer to *e.g.*, Edelman and Jamison [1985] and Korte, Lovász and Schrader [1991]).

Returning to our general model, we denote by $\mathcal{M} = \mathcal{M}(\mathcal{L})$ the collection of all modular functions $m \in \mathbb{R}^{\mathcal{L}}$. $\mathcal{M}$ will always contain the constant function $m_o \in \mathbb{R}^{\mathcal{L}}$, given by

$$m_o(L) \;=\; 1 \quad \text{for all } L \in \mathcal{L} .$$

Furthermore, it is straightforward to verify that $\mathcal{M}$ is a linear subspace of $\mathbb{R}^{\mathcal{L}}$. Given a maximal chain $\mathcal{C}$ in $\mathcal{L}$, we know from Corollary 3.1(b), that the linear map $\iota : \mathcal{M} \to \mathbb{R}^{\mathcal{C}}$, given by

$$\iota(m) \ = \ (m(C))_{C\in\mathcal{C}}$$

is injective. Hence we conclude

$$\dim \mathcal{M} \ \leq \ |\mathcal{C}| \ .$$

We say that the algebraic poset $(\mathcal{L}, \leq, \wedge)$ is *regular* if for every maximal chain $\mathcal{C}$ in $\mathcal{L}$,

$$\dim \mathcal{M}(\mathcal{L}) \ = \ |\mathcal{C}| \ .$$

Thus, in particular, any two maximal chains in a regular algebraic poset have the same cardinality. Moreover, we can find for every vector $y \in \mathbb{R}^{\mathcal{C}}$ of prescribed values on an arbitrary chain $\mathcal{C}$, a modular function $m \in \mathcal{M}$ such that

$$m(C) \ = \ y_C \quad \text{for all } C \in \mathcal{C} \ .$$

Proposition 3.2. *Let $\mathcal{L}$ be a locally distributive closure system on the ground-set E. Then $\mathcal{L}$ is regular.*

Proof. Adjoin a new element $\bar{e}$ to E and set $\bar{E} = E \cup \{\bar{e}\}$. Let

$$\bar{\mathcal{L}} := \{L \cup \{\bar{e}\} \mid L \in \mathcal{L}\} \ .$$

Then $\bar{\mathcal{L}}$ is isomorphic with $\mathcal{L}$. Hence it suffices to prove the Proposition for $\bar{\mathcal{L}}$.

Consider the maximal chain $\bar{\mathcal{C}} = \{\bar{C}_1 \subset \ldots \subset \bar{C}_i \subset \ldots \subset \bar{C}_n = \bar{E}\}$ in $\bar{\mathcal{L}}$. Because $\bar{C}_1 \neq \emptyset$, we can choose elements

$$e_1 \in \bar{C}_1, \ldots, e_i \in \bar{C}_i \setminus \bar{C}_{i-1}, \ldots, e_n \in \bar{E} \setminus \bar{C}_{n-1} \ .$$

We know from Example 3.2 that the indicator functions χ_{e_i}, $i = 1, \ldots, n$, are modular. They are obviously linearly independent. Hence

$$\dim \mathcal{M} \geq |\bar{\mathcal{C}}| \ .$$

$\diamond$

4. Regular Representations

Let $(\mathcal{L}, \leq, \wedge)$ be an algebraic poset as in the previous section and let N be a (finite) set. A *representation of* $(\mathcal{L}, \leq, \wedge)$ *by* N is an injective map

$$\varphi : \mathcal{L} \to 2^N$$

from $\mathcal{L}$ into the collection 2^N of all subsets of N. Note that we do not necessarily require any compatibility of the relations $(\leq, \wedge)$ with the "natural" relations $(\subseteq, \cap)$ on 2^N.

Given the representation φ, every vector $x \in \mathbb{R}^N$ and parameter $r \in \mathbb{R}$ induces a function $x^{(\varphi,r)} : \mathcal{L} \to \mathbb{R}$ via

$$x^{(\varphi,r)}(L) \ := \ x(\varphi(L)) + r \quad \text{for all } L \in \mathcal{L} \ .$$

(Here we use the standard notation $x(\emptyset) := 0$ and $x(S) := \sum_{s \in S} x_s$ for $\emptyset \neq S \subseteq N$.)

We say that the representation $\varphi : \mathcal{L} \to 2^N$ is *regular* if

$$\mathcal{M}(\mathcal{L}) \ = \ \{x^{(\varphi,r)} \mid x \in \mathbb{R}^N, r \in \mathbb{R}\} \ .$$

In other words, a representation by N is regular if we can identify a modular function on $\mathcal{L}$ with a vector in $\mathbb{R}^N$ (up to a constant).

Example 4.1. *Let $\mathcal{L}$ be a closure system with ground-set E. Then the trivial map $\iota(L) = L$ yields a representation of $\mathcal{L}$ by E. Example 3.2 and (the proof of) Proposition 3.2 show that this trivial representation of a closure system is regular if and only if $\mathcal{L}$ is locally distributive.*

We will now discuss another important class of algebraic posets with regular representations.

4.1 Antichains in Partial Orders

Let $P = (N, \preceq)$ be a partial order with ground-set N. We call a subset $A \subseteq N$ an *antichain* (of P) if A does not contain any chain with 2 or more elements. Equivalently, A is an antichain if every two elements in A are incomparable (with respect to P). We denote by $\mathcal{A} = \mathcal{A}(P)$ the collection of all antichains of P. $\mathcal{A}$ has a natural poset structure $(\mathcal{A}, \leq)$ induced by P as follows. For antichains $A, B \in \mathcal{A}$, we set

$$A \leq B \quad \text{if for all } a \in A, \text{ there exists some } b \in B \text{ such that } \ a \preceq b \ .$$

It is well-known that for every two antichains $A, B \in \mathcal{A}$, there exists a unique maximal antichain $A \wedge B$ with $A \wedge B \leq A, B$. $\mathcal{A}$ also possesses a unique maximal element, consisting of the set of all maximal elements of N (relative to P). So $(\mathcal{A}, \leq, \wedge)$ is a lattice and hence an algebraic poset. In fact, it is well-known that a finite lattice $\mathcal{D}$ is distributive if and only if $\mathcal{D}$ is isomorphic with the lattice of antichains relative to some partial order).

We want to give a regular representation of $(\mathcal{A}, \leq, \wedge)$. For every $S \subseteq N$, we define

$$\mathrm{id}(S) := \{e \in N \mid e \preceq s \text{ for some } s \in S\} \ .$$

$S \subseteq N$ is said to be an *ideal* (relative to P) if $S = \mathrm{id}(S)$. Denoting by $\mathrm{MAX}(S)$ the set of maximal elements of a subset $S \subseteq N$ (relative to P), one easily checks that the following holds for every $A, B \in \mathcal{A}$:

(i) $A \leq B$ if and only if $\mathrm{id}(A) \subseteq \mathrm{id}(B)$.
(ii) $A = \mathrm{MAX}(\mathrm{id}(A))$.
(iii) $A \wedge B \ = \ \mathrm{MAX}(\mathrm{id}(A) \cap \mathrm{id}(B))$.

Let $\mathcal{L}(P)$ be the set of ideals of P. Then $(\mathcal{A}, \leq, \wedge)$ is isomorphic with $(\mathcal{L}(P), \subseteq, \cap)$. Because $\mathcal{L}(P)$ is closed under taking unions, we know from Example 4.1 that $\mathcal{L}(P)$ is regular. Hence $\varphi : \mathcal{A} \to \mathcal{L}(P)$, given by

$$\varphi(A) \;=\; \mathrm{id}(A) \;,$$

is a regular representation of $(\mathcal{A}, \leq, \wedge)$. It is important to observe that the trivial representation $\iota : \mathcal{A} \to \mathcal{A}$, given by

$$\iota(A) \;=\; A \;,$$

does not yield a regular representation in general.

Proposition 4.1. *Let $(\mathcal{A}(P), \leq, \wedge)$ be the lattice of antichains relative to the order $P = (N, \preceq)$. Then the trivial representation of $\mathcal{A}(P)$, given by $\iota(A) = A$, is regular if and only if every element $i \in N$ has at most* 1 *upper neighbor relative to P.*

Proof. Suppose there are three elements $i, j, k \in N$ such that both j and k are upper neighbors of i relative to P. We claim that the characteristic function χ_i cannot be modular.

Indeed, consider the antichains $C = \{j, k\}$ and $C' = \mathrm{MAX}(\mathrm{id}(C) \setminus \{k\})$. Let $\mathcal{C}$ be a maximal chain in $\mathcal{A}(P)$ containing both C and C'. Because k and j are upper neighbors of i, we have $i \in \{k\} \wedge \{j\}$ and hence $i \in \{k\} \wedge C'$. Moreover, $j \in C'$ implies $i \notin C'$. With $A = \{k\}$, we therefore obtain for the algebraic triple (A, C', C),

$$-1 = \chi_i(A) - \chi_i(A \wedge C') \;<\; \chi_i(C) - \chi_i(C') = 0 \;,$$

which shows that χ_i is not modular.

Conversely, assume that no $i \in N$ has two upper neighbors relative to P. Consider antichains C and C' such that C is an upper neighbor of C' in $\mathcal{A}(P)$. Then we must have $C' = \mathrm{MAX}(\mathrm{id}(C) \setminus \{k\})$ for a suitable $k \in C$.

Let K' consist of all those elements of N having k as their upper neighbor, *i.e.*, $K' = \mathrm{MAX}(\mathrm{id}(k) \setminus \{k\})$. Because no member of K' has any upper neighbor besides k, we conclude

$$C' \;=\; K' \cup (C \setminus \{k\}) \;.$$

Hence we observe for every $i \in N$,

$$\chi_i(C) - \chi_i(C') \;=\; \begin{cases} \;\;1 & \text{if } i = k \\ -1 & \text{if } i \in K' \\ \;\;0 & \text{otherwise.} \end{cases}$$

Let $A \in \mathcal{A}$ be an antichain such that $A \leq C$ but $A \not\leq C'$. Then $k \in A$ and, as above, we deduce $A \wedge C' = K' \cup A \setminus \{k\}$, which yields

$$\chi_i(A) - \chi_i(A \wedge C') \;=\; \begin{cases} \;\;1 & \text{if } i = k \\ -1 & \text{if } i \in K' \\ \;\;0 & \text{otherwise.} \end{cases}$$

Hence every indicator function χ_i is modular. Since the indicator functions are clearly independent, the Proposition follows.

$\diamond$

We remark that the orders where each element has at most one upper neighbor are the "rooted forests" underlying the computational model in Faigle and Kern [1996]. Queyranne *et al.* [1998] present a model where the underlying order consists of a union of pairwise unrelated chains (and hence is also a rooted forest).

Another important binary operation on the set $\mathcal{A}(P)$ of antichains of the order P has been introduced by Hoffman [1982] (see also Krüger [1997]). For any two antichains $A, B \in \mathcal{A}(P)$, we define the *reduced meet* as

$$A \sqcap B \ := \ (A \wedge B) \cap (A \cup B) \ .$$

Since every subset of an antichain is an antichain, $A \sqcap B$ is an antichain. Moreover, because $A \sqcap B \ \leq \ A \wedge B$ for all $A, B \in \mathcal{A}(P)$, $(\mathcal{A}(P), \leq, \sqcap)$ is a an algebraic poset.

Proposition 4.2. *Let $P = (N, \preceq)$ be an arbitrary order with collection $\mathcal{A}(P)$ of antichains. Then the identity map $\iota(A) = A$ yields a regular representation of the algebraic poset $(\mathcal{A}(P), \leq, \sqcap)$.*

Proof. We show that every incidence function χ_i, $i \in N$, is modular with respect to the reduced meet. We will argue similarly as in the proof of Proposition 4.1.

Let the antichain C be an upper neighbor of the antichain C' in $\mathcal{A}(P)$. Then we have $C' = \mathrm{MAX}(\mathrm{id}(C) \setminus \{k\}$ for some element $k \in C$. Let K' consist of all those elements of $N \setminus \mathrm{id}(C \setminus \{k\})$ having k as their upper neighbor. Then

$$C' \ = \ K' \cup (C \setminus \{k\}) \ .$$

Consider an antichain $A \in \mathcal{A}(P)$ such that $A \leq C$ and $A \not\leq C'$. Then $k \in A$. Let K'' consist of all those elements of $N \setminus \mathrm{id}(A \setminus \{k\})$ having k as their upper neighbor. Because $\mathrm{id}(A) \setminus \{k\} \subseteq \mathrm{id}(C')$, we have

$$A \wedge C' \ = \ K'' \cup (A \setminus \{k\}) \ .$$

Noting $K'' \cap A = \emptyset$ and $K' \subseteq K''$, we have $K'' \cap (A \cup C') = K'' \cap C' = K'$ and hence conclude

$$A \sqcap C' \ = \ K' \cup (A \setminus \{k\}) \ .$$

The modularity of χ_i thus follows as in the previous Proposition.

$\diamond$

5. The Core and the Weber Set

Let $(\mathcal{L}, \leq, \wedge)$ be a non-empty regular algebraic poset and denote by $\mathcal{M} = \mathcal{M}(\mathcal{L})$ the vector space of all modular functions. So $\mathcal{M}$ is a linear subspace of $\mathbb{R}^{\mathcal{L}}$.

Let $f \in \mathbb{R}^{\mathcal{L}}$ be given. We associate with f the polyhedron

$$\mathcal{P}(f) \;:=\; \{m \in \mathcal{M} \mid m \leq f\} .$$

In the case $f = 0$, we call $\mathcal{P}(0)$ the *modular recession cone* of $\mathcal{L}$.

We say that a modular function $m \in \mathcal{M}$ is *Weber* (relative to f) if there exists a maximal chain $\mathcal{C} \subseteq \mathcal{L}$ such that

$$m(C) \;=\; f(C) \quad \text{for all } C \in \mathcal{C} .$$

The *Weber set* $\mathcal{W}(f)$ is then defined as the convex hull of all (modular) Weber functions (relative to f). Because $\mathcal{L}$ is regular (by assumption), we have $\mathcal{W}(f) \neq \emptyset$ for all $f \in \mathbb{R}^{\mathcal{L}}$.

In order to formulate our main result in this section, we have to introduce a special property $(\mathcal{L}, \leq, \wedge)$ is required to have. We say that $(\mathcal{L}. \leq, \wedge)$ has the *modular chain property* if the following is true:

(MC) For every $\rho \in \mathcal{V}_+^*$, there exists some $\rho^* \in \mathcal{V}_+^*$ such that
- (i) $\rho^*(m) = \rho(m)$ for all $m \in \mathcal{M}$;
- (ii) $\mathrm{supp}(\rho^*)$ is a chain in $\mathcal{L}$.

(Recall here the notation $\mathcal{V}_+ = \mathbb{R}_+^{\mathcal{L}}$ *etc.* introduced in Section 2.)

Theorem 5.1. *Let $(\mathcal{L}, \leq, \wedge)$ be a regular algebraic poset satisfying the modular chain property (MC). Then for every $f \in \mathbb{R}^{\mathcal{L}}$,*

$$\mathcal{P}(f) \;\subseteq\; \mathcal{W}(f) + \mathcal{P}(0) .$$

Proof. Given the modular function $x \in \mathcal{P}(f)$, we have to show that there exist modular functions $w \in \mathcal{W}(f)$ and $y \in \mathcal{P}(0)$ such that $x = w + y$. Noting $\mathcal{W}(f - x) = \mathcal{W}(f) - x$, and replacing f by $f' = f - x$, if necessary, we can apparently assume without loss of generality that $x = 0$ and hence $f \geq 0$ (since $0 \in \mathcal{P}(f)$).

Suppose $0 \notin \mathcal{W}(f) + \mathcal{P}(0)$. We will derive a contradiction.

Because $\mathcal{K} := \mathcal{W}(f) + \mathcal{P}(0)$ is the Minkowski sum of two polyhedra, $\mathcal{K}$ is itself a polyhedron. Hence $0 \notin \mathcal{K}$ means that we can find a hyperplane in $\mathbb{R}^{\mathcal{L}}$ so that $\mathcal{K}$ is contained in one of the associated open halfspaces, *i.e*, there exists a linear functional $\gamma : \mathbb{R}^{\mathcal{L}} \to \mathbb{R}$ such that

$$\gamma(w + y) \;>\; 0 \quad \text{for all } w \in \mathcal{W}(f),\, y \in \mathcal{P}(0) .$$

Since $0 \in \mathcal{P}(0)$, we thus obtain, in particular, $\gamma(w) > 0$ for all $w \in \mathcal{W}(f)$.

Consider the linear program (P):

$$\begin{array}{lcl} \max & -\gamma(m) & \\ s.t. & m & \in\ \mathcal{P}(0) \end{array}$$

Because (P) is feasible and bounded (by $\gamma(w)$, for example), (P) has an optimal solution. By linear programming duality, we therefore know that the dual linear program (D)

$$\begin{array}{lrcll} \min & \rho(0) & & & \\ s.t. & \rho & \in & \mathcal{V}_+^* & \\ & \rho(m) & = & -\gamma(m) & \text{for all } m \in \mathcal{M} \end{array}$$

has a feasible solution ρ^*. In view of the modular chain property (MC), we may assume that $\mathrm{supp}(\rho^*)$ is a chain in $\mathcal{L}$.

Let $\mathcal{C}$ be a maximal chain containing $\mathrm{supp}(\rho^*)$ and let w^* be the unique modular function satisfying $w^*(C) = f(C)$ for all $C \in \mathcal{C}$. Then $w^* \in \mathcal{W}(f)$. In view of $f \geq 0$ and $\rho^* \in \mathcal{V}_+^*$, Lemma 2.2 now yields

$$-\gamma(w^*) \;=\; \rho^*(f) \;\geq\; 0\;,$$

which implies $\gamma(w^*) \leq 0$, contradicting $\gamma(w) > 0$ for all $w \in \mathcal{W}(f)$.

$\diamond$

Corollary 5.1. *Let $(\mathcal{L}, \leq, \wedge)$ be a regular algebraic poset satisfying the modular chain condition (MC) and let $f \in \mathbb{R}^{\mathcal{L}}$ be given. Then*

$$\mathcal{P}(f) \;=\; \mathcal{W}(f) + \mathcal{P}(0)$$

if and only if f is submodular.

Proof. Since $\mathcal{P}(0)$ is the recession cone of $\mathcal{P}(f)$, we always have

$$\mathcal{P}(f) \;=\; P(f) + \mathcal{P}(0)\;.$$

Assume that f is submodular. From Corollary 3.1, then, we conclude $g \leq f$ for every $g \in \mathcal{W}(f)$, *i.e.*, $W(f) \subseteq \mathcal{P}(f)$, and $\mathcal{P}(f) \supseteq \mathcal{W}(f) + \mathcal{P}(0)$ follows.

Conversely, assume that f is not submodular. Then Theorem 3.2 guarantees the existence of some $w \in \mathcal{W}(f)$ such that $w \notin \mathcal{P}(f)$. So $w = w + 0 \in W(f) + \mathcal{P}(0)$ shows $\mathcal{P}(f) \neq \mathcal{W}(f) + \mathcal{P}(0)$.

$\diamond$

We will now exhibit a sufficient condition on $\mathcal{L}$, essentially due to Hoffman [1982], that implies the modular chain condition (MC).

Assume that $(\mathcal{L}, \leq)$ is equipped with two binary operations $\oplus$ and $\otimes$ such that for all incomparable elements $A, B \in \mathcal{L}$, $A \oplus B \in \mathcal{L}$ and $A \otimes B \in \mathcal{L}$ are defined and satisfy

$$A \;<\; A \oplus B \quad \text{and} \quad B \;<\; A \oplus B\;.$$

Assume furthermore that each $m \in \mathcal{M}$ is globally modular relative to $\oplus$ and $\otimes$, *i.e.*,

$$m(A \oplus B) + m(A \otimes B) \;=\; m(A) + m(B)\;.$$

(Note that we do not require any compatibility of $\oplus$ and $\otimes$ with the operation $\wedge$ on $\mathcal{L}$).

Proposition 5.1. *Let $(\mathcal{L}, \leq)$ be a poset and $\mathcal{M} \subseteq \mathbb{R}^{\mathcal{L}}$ be a linear subspace of real-valued functions on $\mathcal{L}$ such that every $m \in \mathcal{M}$ is globally modular with respect to $\oplus$ and $\otimes$. Then the modular chain condition (MC) holds for $\mathcal{M}$.*

Proof. Fix a "linear extension" π of $(\mathcal{L}, \leq)$, *i.e.*, a linear ordering π of the members of $\mathcal{L}$ such that A occurs before B whenever $A > B$ holds relative to $(\mathcal{L}, \leq)$.

Let $\rho \in \mathcal{V}_+^*$ be given and choose $\rho^* \in \mathcal{V}_+^*$ with lexicographically maximal support relative to the linear extension π under the condition $\rho^*(m) = \rho(m)$ for all $m \in \mathcal{M}$. We claim that $\text{supp}(\rho^*)$ is a chain in $\mathcal{L}$.

Suppose that this is not the case and that we can find incomparable elements $A, B \in \text{supp}(\rho^*)$ with $\rho_A^* = \rho^*(\chi_A) > 0$ and $\rho_B^* = \rho^*(\chi_B) > 0$, where χ denotes the respective indicator function.

Choose $0 < \epsilon \leq \min\{\rho_A^*, \rho_B^*\}$ and define $\rho' \in \mathcal{V}^*$ via

$$\rho'_L \quad = \quad \begin{cases} \rho_L^* + \epsilon & \text{if } L = A \oplus B \text{ or } L = A \otimes B \\ \rho_L^* - \epsilon & \text{if } L = A \text{ or } L = B \\ \rho_L^* & \text{otherwise.} \end{cases}$$

Then $\rho' \in \mathcal{V}_+^*$. Moreover, because $m(A \oplus B) + m(A \otimes B) = m(A) + m(B)$ holds for all $m \in \mathcal{M}$, we have

$$\rho'(m) = \rho^*(m) \quad \text{for all } m \in \mathcal{M} \; .$$

On the other hand, $\rho'_{A \oplus B} > \rho^*_{A \oplus B}$ exhibits ρ' as lexicographically greater than ρ^* relative π ($A \oplus B$ occurs before A and B in π!), which contradicts the choice of ρ^*.

◇

Corollary 5.2. *Let $(\mathcal{L}, \subseteq, \cap)$ be a closure system on E that is closed under union and intersection. Then for every $f : \mathcal{L} \to \mathbb{R}$,*

$$\mathcal{P}(f) \;\subseteq\; \mathcal{W}(f) + \mathcal{P}(0) \; .$$

Proof. The indicator functions χ_e are globally modular with respect to $\cup$ and $\cap$. Therefore, by Proposition 5.1, $(\mathcal{L}, \subseteq, \cap)$ satisfies the modular chain condition (MC). Hence the claim follows from Theorem 5.1.

◇

In the case where $\mathcal{L}$ consists of all order ideals relative to an order $P = (E, \leq)$ on the ground-set E, Corollary 5.2 yields Theorem 3.5 of Derks and Gilles [1995].

For a given closure system $(\mathcal{L}, \subseteq, \cap)$ on the ground-set E and a function $f : \mathcal{L} \to \mathbb{R}$, the notion of *core* is an important solution concept in cooperative game theory that goes back to von Neumann and Morgenstern [1944]. For our purposes, we can define the (modular) core of f as

$$\text{core}(f) \;:=\; \{x \in \mathcal{P}(f) \,|\, x(E) = f(E)\} \; .$$

Our analysis thus yields the following result, due to Weber [1988]:

Corollary 5.3. *Consider* $(2^E, \subseteq, \cap)$. *Then* $core(f) \subseteq \mathcal{W}(f)$ *holds for every* $f : 2^E \to \mathbb{R}$ *satisfying* $f(\emptyset) = 0$.

Proof. Consider $x \in \mathrm{core}(f)$. By Corollary 5.2, we can find $w \in \mathcal{W}(f)$ and $y \in \mathcal{P}(0)$ such that $x = w + y$.

Since $x(E) = f(E) = w(E)$, we must have $y(E) = 0$. So y must be 0, and $x = w$ follows.

◇

Corollary 5.1 may fail to hold for closure systems that are not union-closed. The next observation is due to Jiménez [1998] in the context of convex geometries.

Corollary 5.4. *Let* $(\mathcal{L}, \subseteq, \cap)$ *be a locally distributive closure system on* E *and* $f \in \mathbb{R}^{\mathcal{L}}$ *a given function. Then* $\mathcal{W}(f) \subseteq core(f)$ *holds if and only if* f *is submodular.*

Proof. Theorem 3.1.

◇

6. The Monge and the Greedy Algorithm

We assume now that we are given a poset $(\mathcal{L}, \leq)$ and a linear subspace $\mathcal{M} \subseteq \mathbb{R}^{\mathcal{L}}$ together with a linear map $\gamma : \mathcal{M} \to \mathbb{R}$. Recalling the notation $\mathcal{V}^*$ for the dual of $\mathcal{V} = \mathbb{R}^{\mathcal{L}}$ from Section 2, our goal is to find some $\rho \in \mathcal{V}_+^*$ such that

(i) $\rho(m) = \gamma(m)$ for all $m \in \mathcal{M}$.
(ii) $\mathrm{supp}(\rho)$ is a chain in $\mathcal{L}$,

whenever such a ρ exists at all. In order to formulate our algorithm for solving this problem, we make the further assumptions (M_0)-(M_3):

(M_0) We can find a basis $\{m_1, \ldots, m_n\}$ of $\mathcal{M}$ and non-negative coefficients $m_i(L) \geq 0$ such that for $i = 1, \ldots, n$,

$$m_i = \sum_{L \in \mathcal{L}} m_i(L) \chi_L \ .$$

(M_1) For all $L, L' \in \mathcal{L}$ with $L < L'$, there exists some $1 \leq i \leq n$ such that $m_i(L') \neq 0$ and $m_i(L) = 0$.

Given such a basis $\{m_1, \ldots, m_n\} \subseteq \mathcal{V}_+$, we set $N = \{1, 2, \ldots, n\}$ and define for every $J \subseteq N$,

$$\mathcal{L}_J \ := \ \{L \in \mathcal{L} \mid m_j(L) = 0 \text{ for all } j \in J\} \ .$$

(M_2) For every $J \subseteq N$, either $\mathcal{L}_J = \emptyset$ or the subset $\mathcal{L}_J \subseteq \mathcal{L}$ has a unique maximal element $\mathbf{1}_J$.

In particular, we assume with (M_2) that the poset $(\mathcal{L}, \leq)$ possesses a unique maximal element $\mathbf{1} = \mathbf{1}_\emptyset$.

(M_3) For every chain $A < B < C$ in $\mathcal{L}$ and $1 \leq i \leq n$,
$m_i(A) \neq 0 \neq m_i(C)$ implies $m_i(B) \neq 0$.

(Note that property (M_3) can be viewed as a generalized "consecutive 1's property" of $\mathcal{M}$ relative to $\mathcal{L}$).

Lemma 6.1. *Let $\mathcal{M} \subseteq \mathcal{L}$ satisfy the properties (M_0)-(M_3). Then*

$$\dim \mathcal{M} \quad \geq \quad \begin{cases} |\mathcal{C}| & \text{if } \mathcal{L}_N = \emptyset \\ |\mathcal{C}| - 1 & \text{if } \mathcal{L}_N \neq \emptyset. \end{cases}$$

In particular, if $\mathcal{M}$ is the vector space of modular functions on the algebraic poset $\mathcal{L} = (\mathcal{L}, \leq, \wedge)$, then $\mathcal{L}$ is regular.

Proof. Assume $\mathcal{L}_N \neq \emptyset$. In view of (M_1), $\mathbf{1}_N$ must be a minimal element of $\mathcal{L}$. (M_2), moreover, shows that $\mathbf{1}_N$ is the unique minimal element. Consider now the chain

$$C_1 \; < \; C_2 \; < \; \ldots \; < \; C_m$$

in $\mathcal{L}$. If $C_1 \neq \mathbf{1}_N$, (M_1) and (M_3) guarantee the existence of elements $i_1, i_2, \ldots, i_m \in N$ such that for all $k = 1, \ldots, m$,

$$m_{i_k}(C_k) \neq 0 \;\text{ and }\; m_{i_k}(C_j) = 0 \;\text{ whenever } j < k \; .$$

Hence we deduce $\dim \mathcal{M} \geq m$. If $C_1 = \mathbf{1}_N$, we apply the same argument to the chain $C_2 < C_3 < \ldots < C_m$.

If $\mathcal{M}$ consists of the modular functions on $(\mathcal{L}, \leq, \wedge)$, then $\mathcal{L}_N = \emptyset$ (because, *e.g.*, the constant function $m_0 = 1$ is a member of $\mathcal{M}$). So we have for every maximal chain $\mathcal{C} \subseteq \mathcal{L}$,

$$|\mathcal{C}| \; \leq \; \dim \mathcal{M} \; \leq \; |\mathcal{C}| \; .$$

$\diamond$

Example 6.1. *Let $(\mathcal{L}, \subseteq)$ be a system of subsets of the ground-set N such that $N \in \mathcal{L}$ and $\mathcal{L}$ is closed under taking unions. Let $\mathcal{M}$ be the subspace of $\mathbb{R}^{\mathcal{L}}$ that is generated by the indicator functions $m_i = \chi_i$, $i \in N$. Then*

$$m_i(L) = \begin{cases} 1 & \text{if } i \in L \\ 0 & \text{if } i \notin L, \end{cases}$$

and it is straightforward to verify the properties(M_1)-(M_3) as well. For $J \subseteq N$, $\mathbf{1}_J$ is the unique maximal member of $\mathcal{L}$ having an empty intersection with J. (One may obtain such systems, for examples, from "antimatroids", i.e., the systems of the complements of the closed sets in convex geometries (cf. Korte et al. [1991] for details)).

Similarly, let $(\mathcal{A}, \leq)$ be the poset of antichains of the ordered set $P = (N, \preceq)$. Taking the subspace $\mathcal{M} \subseteq \mathbb{R}^{\mathcal{A}}$ again to be generated by the indicator functions χ_i, $i \in N$, it is easy to see that the properties (M_0)-(M_3) hold.

For $J \subseteq N$, consider the unique largest ideal I relative to P containing none of the elements in J. Then $\mathbf{1}_J$ is the antichain of maximal elements of I.

$\diamond$

In view of property (M_0) and with the notation $c_i := \gamma(m_i)$, $i \in N$, our goal is now to find a non-negative vector $\rho \in \mathbb{R}^{\mathcal{L}}$ such that

(i') For all $i \in N$,

$$\sum_{\mathcal{L} \in \mathcal{L}} \rho_L \cdot m_i(L) \;=\; c_i \;.$$

(ii') $\mathrm{supp}(\rho)$ is a chain in $\mathcal{L}$.

The following algorithm generalizes the "Dual Greedy Algorithm" of Faigle and Kern [1997], which in turn generalizes the classical algorithm of Monge [1781] (see also Burkard *et al.* [1996] and Hoffman [1985]).

Monge Algorithm:

Initialize:

$\rho_L \leftarrow 0$ for all $L \in \mathcal{L}$;
$w_i \leftarrow c_i$ for all $i \in N$;
$J \leftarrow \emptyset$;
$T \leftarrow \mathbf{1}$;

Iterate:

WHILE $\mathcal{L}_J \neq \emptyset$ DO:
Determine some i such that $m_i(T) \neq 0$ and $w_i/m_i(T)$ minimal;
$\rho_T \leftarrow [w_i/m_i(T)]$;
$w_j \leftarrow [w_j - \rho_T \cdot m_j(T)]$ for all $j \in N$;
$J \leftarrow [J \cup \{i\}]$;
$T \leftarrow \mathbf{1}_J$;

Our next result states that the Monge algorithm will always find a feasible solution whose support is a chain, provided such a solution exists at all. Moreover, this solution is unique.

Theorem 6.1. *Assume that the properties (M_0)-(M_3) hold and let $\rho \in \mathbb{R}^{\mathcal{L}}_+$ be a vector satisfying*

(a) *For all $i \in N$, $\sum_{L \in \mathcal{L}} \rho_L \cdot m_i(L) = c_i$.*

(b) *$\mathrm{supp}(\rho)$ is a chain in $\mathcal{L}$.*

Let $\rho^* \in \mathbb{R}^{\mathcal{L}}$ be the vector computed by the Monge algorithm. Then $\rho^* = \rho$.

Proof. We prove the Theorem by induction on the dimension $n = |N|$ of the subspace $\mathcal{M}$ of $\mathbb{R}^{\mathcal{L}}$ under the assumption that there exists a non-negative vector $\rho \in \mathbb{R}^{\mathcal{L}}_+$ satisfying the feasibility conditions (a) and (b) above. We denote the support of ρ by

$$\mathrm{supp}(\rho) \;=\; \{C_1 < \ldots < C_m\} \;.$$

Consider first the situation with $n = 1$. Because $n = \dim \mathcal{M} \geq |\mathcal{C}| - 1$ for every chain $\mathcal{C} \subseteq \mathcal{L}$ (Lemma 6.1), we see that $\mathcal{L}$ cannot admit any chain of 3 or more elements. Since $\mathcal{L}$ possesses a unique maximal element $\mathbf{1} = \mathbf{1}_\emptyset$ and $\mathbf{1}_N$ is the unique minimal element of $\mathcal{L}$ if $\mathcal{L}_N \neq \emptyset$, $n = 1$ implies that $\mathcal{L}$ is a single chain with at most two elements. So the statement of the Theorem is clearly true.

Assume now that the Theorem holds for all $n - 1 \geq 1$ and consider the maximal element $\mathbf{1}$ of $\mathcal{L}$. We consider two cases:

Case 1: $C_m \neq \mathbf{1}$.

By properties (M_1) and (M_3), there exists some $i \in N$ such that $m_i(\mathbf{1}) \neq 0$ and $m_i(C_k) = 0$ for $k = 1, \ldots, m$. Hence we have $\rho_{\mathbf{1}} = 0$ and, consequently, $c_i = 0$. Thus we conclude $\rho_{\mathbf{1}} = \rho^*_{\mathbf{1}} = 0$. Moreover, we must have

$$\mathrm{supp}(\rho) \;\subseteq\; \mathcal{L}_j$$

for every $j \in N$ that could be selected by the Monge Algorithm in the first iteration.

By induction, ρ is uniquely determined by the Monge Algorithm relative to $\mathcal{L}' = \mathcal{L}_i$ and $\mathcal{M}' = \mathcal{M} \cap \mathbb{R}^{\mathcal{L}'}$, and the Theorem follows.

Case 2; $C_m = \mathbf{1}$.

Similarly to Case 1, there exists some $i \in N$ such that $m_i(\mathcal{C}_m) \neq 0$ but $m_i(C_k) = 0$ for all $k < m$, which implies

$$\rho_{\mathbf{1}} \;=\; c_i / m_i(\mathbf{1}) \; .$$

Because ρ is feasible, we have $\rho_{\mathbf{1}} \cdot m_j(\mathbf{1}) \leq c_j$ for all $j \in N$. So the Monge Algorithm will compute $\rho^*_{\mathbf{1}} = \rho_{\mathbf{1}}$. Adjusting the parameters for all $j \in N$ to

$$c'_j \;:=\; c_j - \rho_{\mathbf{1}} \cdot m_j(\mathbf{1}) \; ,$$

and denoting by ρ' the restriction of ρ to $\mathcal{L} \setminus \{\mathbf{1}\}$, we see that ρ' satisfies for all $j \neq i$:

$$\sum_{L \neq \mathbf{1}} \rho'_L \cdot m_j(L) = c'_j \; .$$

By induction, we therefore conclude as before that $\rho = \rho^*$ holds.

$\diamond$

We now assume that $(\mathcal{L}, \leq, \wedge)$ is an algebraic poset and $\mathcal{M} \subseteq \mathbb{R}^{\mathcal{L}}$ the associated vector space of modular functions. Given the linear map $\gamma : \mathcal{M} \to \mathbb{R}$ and the function $f \in \mathbb{R}^{\mathcal{L}}$, we consider the optimization problem

$$\begin{array}{llcll} \text{(P)} & \max & \gamma(m) & & \\ & s.t. & m & \leq & f \\ & & m & \in & \mathcal{M} \, . \end{array}$$

Assume, furthermore, that $\mathcal{M}$ satisfies the conditions (M_0)-(M_3). Then we can try to solve the problem with the following 2-phase procedure:

Greedy Algorithm:

Phase 1:

Use the Monge Algorithm to find a $\rho^* \in \mathbb{R}_+^{\mathcal{L}}$ such that
(i) $\rho^*(m) = \gamma(m)$ for all $m \in \mathcal{M}$.
(ii) $\mathrm{supp}(\rho^*)$ is a chain in $\mathcal{L}$.

Phase 2:

(a) Determine a maximal chain $\mathcal{C} \subseteq \mathcal{L}$ so that $\mathrm{supp}(\rho^*) \subseteq \mathcal{C}$.
(b) Determine a modular function $m^* \in \mathcal{M}$ so that
$m^*(C) = f(C)$ for all $C \in \mathcal{C}$.

Theorem 6.2. *Assume that the algebraic poset $(\mathcal{L}, \leq, \wedge)$ satisfies the conditions (M_0)-(M_3). Assume furthermore that both phases of the Greedy Algorithm produce feasible solutions. Then the solutions m^* computed in Phase 2 is optimal for (P). Moreover, the solution ρ^* computed in Phase 1 is optimal for the problem (D):*

$$\begin{array}{lrcl} \min & \rho(f) & & \\ s.t. & \rho & \in & \mathbb{R}_+^{\mathcal{L}} \\ & \rho(m) & = & \gamma(m) \quad \textit{for all } m \in \mathcal{M}. \end{array}$$

Proof. The Theorem is a direct consequence of the complementary slackness property of linear programs (Lemma 2.2).

◇

In view of Theorem 6.1, we know that Phase 1 of the Greedy Algorithm will be successful if $\mathcal{L}$ satisfies the modular chain condition (MC) (*cf.* Section 5). Given that Phase 1 produces a feasible ρ^*, Phase 2 will be successful, for example, if f is submodular (Corollary 3.1). The 2-phase greedy algorithms of Kornblum [1978] or Faigle and Kern [1996, 1997], for example, are special cases of this Greedy Algorithm with respect to suitable regular representations of certain algebraic posets $(\mathcal{L}, \leq, \wedge)$.

6.1 Systems of Antichains

Let $P = (N, \preceq)$ be an ordered set with collection $\mathcal{A} = \mathcal{A}(P)$ of antichains and collection $\mathcal{D} = \mathcal{D}(P)$ of ideals. Since the antichains of P are in a one-to-one correspondence with the ideals of P, we can identify $\mathbb{R}^{\mathcal{A}}$ with $\mathbb{R}^{\mathcal{D}}$. More explicitly, $(\mathcal{A}, \leq, \wedge)$ and $(\mathcal{D}, \subseteq, \cap)$ are naturally isomorphic via the map $h : \mathcal{A} \to \mathcal{D}$, where for all $A \in \mathcal{A}$,

$$h(A) \;=\; \mathrm{id}(A)$$

with inverse $h^{-1}(D) = \mathrm{MAX}(D)$. In particular, every modular map $m : \mathcal{A} \to \mathbb{R}$ can be interpreted as a modular map $m' : \mathcal{D} \to \mathbb{R}$ via for all $D \in \mathcal{D}$,

$$m'(D) \;=\; m(\mathrm{MAX}(D)) \;,$$

and conversely. Similarly, the linear functional γ on the vector space $\mathcal{M}$ of modular functions on $\mathcal{A}$ can be identified with a linear functional γ' with respect to $\mathcal{D}$ via

$$\gamma'(m') \;=\; \gamma(m)\;.$$

Hence the optimization problem

$$\begin{array}{llrcl} \text{(P)} & \max & \gamma(m) & & \\ & s.t. & m & \leq & f \\ & & m & \in & \mathcal{M} \end{array}$$

is equivalent with the problem

$$\begin{array}{llrcl} \text{(P')} & \max & \gamma'(m') & & \\ & s.t. & m & \leq & f \\ & & m & \in & \mathcal{M} \end{array}$$

Our point here is the observation that the linear optimization problem (P) relative to modular functions on *antichains* is essentially the same as the linear optimization problem (P') relative to modular functions on *ideals*. We may solve either one in order to obtain a solution for the other.

If P is a rooted forest, then Proposition 4.2 shows that both the trivial representation of $(\mathcal{A}, \leq, \wedge)$ and the trivial representation of $(\mathcal{D}, \subseteq, \cap)$ are regular. Assuming $f(\emptyset) = 0$ for the simplicity of the discussion, (P) can be written as the linear program

$$\begin{array}{llrcll} \text{(RP)} & \max & \sum\limits_{i\in N} c_i x_i & & & \\ & s.t. & \sum\limits_{i\in A} x_i & \leq & f(A) & \text{for all } A \in \mathcal{A} \\ & & x & \in & \mathbb{R}^N, & \end{array}$$

where $c_i = \gamma(\chi_i)$, with χ_i being the incidence function of $i \in N$ relative to the antichains.

Similarly, (P') can be written as the linear program

$$\begin{array}{llrcll} \text{(RP')} & \max & \sum\limits_{i\in N} c'_i x_i & & & \\ & s.t. & \sum\limits_{i\in \mathrm{id}(A)} x_i & \leq & f(A) & \text{for all } A \in \mathcal{A} \\ & & x & \in & \mathbb{R}^N & \end{array}$$

for suitable coefficients c'_i. The representation by ideals represents modular functions as linear combinations of the incidence functions X_i of the elements $i \in N$ relative to the ideals. So we have for all $i \in N$,

$$c'_i \;=\; \gamma(X_i)\;.$$

How do the c'_i in (RP') relate to the c_i in (RP)? If $P = (N, \preceq)$ is a rooted forest, we have for all $i \in N$,

$$X_i = \sum_{j \succeq i} \chi_j \; .$$

Hence

$$c'_i \; = \; \gamma\left(\sum_{j \succeq i} \chi_j\right) \; = \; \sum_{j \succeq i} \gamma(\chi_j) \; = \; \sum_{j \succeq i} c_j \; .$$

We mention some applications.

Notice that the representation of antichains by ideals and modular functions $m \in \mathcal{M}$ by vectors $x \in \mathbb{R}^N$ allows us to reduce the model of modular functions on distributive lattices with submodular constraints of Danilov *et al.* [1999] to that of classical "submodular systems" (*cf.* Fujishige [1991]), which in turn can be studied within the framework of classical matroid theory (*cf.* Faigle [1987]).

In the case of rooted forests, the optimization problem (RP) can be solved in the form (RP'). For submodular f, our construction reduces the optimization models of Queyranne *et al.* [1998] and of Faigle and Kern [1996] to classical matroid theory. The greedy algorithms there can thus be viewed as manifestations of the greedy algorithm for matroids (on unordered groundsets). In particular, our construction provides a matroidal model for the bipartite assignment problem with Monge-costs on the edges (*i.e.*, the case where $P = (N, \preceq)$ is the union of two unrelated chains and f is submodular).

In addition, we can now also solve the optimization problem (RP) for rooted forests if f is given as the minimum of two submodular functions: rewriting the problem in the form (RP'), we may apply Edmonds' [1979] matroid intersection algorithm, for example (*cf.* also Fujishige [1991] for details on the latter).

For general $P = (N, \preceq)$, however, we do not know of an efficient algorithm to solve (RP) even when f is submodular with respect to the algebraic poset $(\mathcal{A}(P), \leq, \wedge)$.

If f is submodular with respect to $(\mathcal{A}(P), \leq, \sqcap)$, the Greedy Algorithm above will solve (RP). However, we do not know whether (RP) can be reduced to an analogous optimization problem on an *unordered* ground-set. In particular, we do not know how to solve (RP) efficiently for general P if f is given as the minimum of two functions that are submodular relative to $(\mathcal{A}(P), \leq, \sqcap)$.

References

[1] J.M. Bilbao [1998]: *Closure spaces and restricted games.* To appear in: Mathematical Methods of Operations Research.

[2] G. Birkhoff [1967]: *Lattice Theory.* 3rd ed., A.M.S. Colloquium Publ. XXV, Providence, R.I..

[3] R.E. Burkard, B. Klinz, and R. Rudolf [1996]: *Perspectives of Monge properties in optimization.* Discrete Applied Mathematics 70, 95-161.

[4] V.I. Danilov, G.A. Koshevoy, and A.I. Sotskov [1999]: *Equilibrium in an Economy of Information Goods.* To appear in: *Markets, Information and Uncertainty: Essays in Economic Theory in honor of Kenneth Arrow*, G. Chichilnisky, ed., Cambridge University Press, Cambridge.

[5] J.J.M. Derks and R.P. Gilles [1995]: *Hierarchical organization structures and constraints on coalition formation.* International Journal of Game Theory 24, 147-163.

[6] P.H. Edelman and R.E. Jamison [1985]: *The theory of convex geometries.* Geometriae Dedicata 19, 247-270.

[7] J. Edmonds [1970]: *Submodular functions, matroids and certain polyhedra.* In *Combinatorial Structures and Their Applications*, R. Guy *et al.*, eds., Gordon and Breach, New York, 69-87.

[8] J. Edmonds [1979]: *Matroid intersection.* Annals of Discrete Mathematics 4, 39-49.

[9] U. Faigle [1987]: *Matroids in Combinatorial Optimization.* In: *Combinatorial Geometries*, N. White, ed., Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 161-210.

[10] U. Faigle and W. Kern [1992]: *The Shapley value for cooperative games under precedence constraints.* International Journal of Game Theory 21, 249-266.

[11] U. Faigle and W. Kern [1996]: *Submodular linear programs on forests.* Mathematical Programming 72, 195-206.

[12] U. Faigle and W. Kern [1997]: *On the core of submodular cost games.* Preprint, University of Twente. To appear: Mathematical Programming.

[13] A. Frank [1998]: *Increasing the rooted-connectivity of a digraph by one.* Preprint, Eötvös University. To appear: Mathematical Programming.

[14] S. Fujishige [1991]: *Submodular Functions and Optimization.* Annals of Discrete Mathematics, vol. 47, Elsevier, Amsterdam.

[15] A.J. Hoffman [1982]: *Ordered sets and linear programming.* In: *Ordered Sets*, I. Rival, ed., Reidel, 619-654.

[16] A.J. Hoffman [1985]: *On greedy algorithms that succeed.* In: *Surveys in combinatorics*, Cambridge University Press, 97-112.

[17] N. Jiménez [1998]: *Solution Concepts for Games over Closure Spaces.* Ph.D.-Thesis, University of Seville, Seville.

[18] D. Kornblum [1978]: *Greedy algorithms for some optimization problems on a lattice polyhedron.* Ph.D. thesis, Graduate Center of the City University of New York.

[19] B. Korte, L. Lovász, and R. Schrader [1991]: *Greedoids.* Springer-Verlag, Berlin.

[20] U. Krüger [1997]: *Structural aspects of ordered polymatroids.* Preprint. To appear: Discrete Applied Mathematics.

[21] G. Monge [1781]:*Déblai et Remblai.* Mém. de L'Academie des Sciences.

[22] J. von Neumann and O. Morgenstern [1944]: *Theory of Games and Economic behavior.* Princeton University Press, Princeton, 1944, 1947.

[23] M. Queyranne, F. Spieksma, and F. Tardella [1998]: *A general class of greedily solvable linear programs.* To appear: Mathematics of Operations Research.

[24] R.J. Weber [1988]: *Probabilistic values for games.* In: *The Shapley Value*, A.E. Roth, ed., Cambridge University Press, Cambridge.

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