NOTE ON THE COMPUTATIONAL COMPLEXITY OF LEAST CORE CONCEPTS FOR MIN-COST SPANNING TREE GAMES

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Abstract. Various least core concepts including the classical least core of cooperative games are discussed. By a reduction from minimum cover problems, we prove that computing an element in these least cores is in general NP-hard for minimum cost spannning tree games. As a consequence, computing the nucleolus, the nucleon and the per-capita nucleolus of minimum cost spanning tree games is also NP-hard.

1. Introduction

Minimum cost spanning tree problems have been widely studied in the literature. After their introduction by Bird [1976], various results about the core and nucleolus were established (see, *e.g.*, Aarts [1994], Granot and Huberman [1981], [1984]).

In this note, we discuss the least core of a cooperative game (see Maschler *et al.* [1979]) and several variants of this solution concept. We prove that computing an allocation according to these least core concepts is in general NP-hard for minimum cost spanning tree games. It was shown in Faigle *et al.* [1998b] that computing the nucleolus of minimum cost spanning tree games is NP-hard. We obtain this result as an immediate corollary from our main result. Furthermore, we are able to show that computing other solution concepts such as the nucleon (cf. Faigle *et al.* [1998a]) of minimum cost spanning tree games is NP-hard.

A *cooperative game* is described by a pair (N, c), where N is a finite set of n players and $c : 2^N \to \mathbb{R}^+$ is a *cost function* satisfying $c(\emptyset) = 0$. A *coalition* is a subset $S \subseteq N$. $c(S)$ is called the *cost* of coalition S with the interpretation that $c(S)$ is the joint cost of the players in S if they decide to cooperate.

A central problem in cooperative game theory is to find a 'fair' allocation of the total costs $c(N)$ to the players. A vector $x \in \mathbb{R}^N$ is an *allocation* if $x(N) = c(N)$. (Throughout the paper, we use the shorthand notation $x(S) = \sum_{i \in S} x_i$.)

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The idea of the *core* of a game essentially goes back to von Neumann and Morgenstern [1944]. core(c) is the set of all allocations x for which there is no coalition $S \subseteq N$ such that $x(S) > c(S)$, which means that no coalition should have to pay more than its cost.

There are games for which core(c) is empty. The *least core* of a game attempts to maximize the satisfaction $c(S) - x(S)$ over all coalitions $S \neq \emptyset, N$. leastcore(c) is defined to consist of all optimal solutions x for the linear program

$$\begin{array}{llcll} \max & \epsilon & & & \\ s.t & x(S) & \leq & c(S) - \epsilon & \text{for all } S \not\subseteq N, S \neq \emptyset \\ & x(N) & = & c(N). & \end{array}$$

It is not hard to see that leastcore(c) is non-empty.

A *minimum cost spanning tree* game (MCST-game, for short) is defined by a set N of players, a *supply* node $s \notin N$, a complete graph with vertex set $V = N \cup \{s\}$ and by a non-negative *distance* or *length* function $l \geq 0$ defined on the edge set of the complete graph. The *cost* $c(S)$ of a coalition $S \subseteq N$ is, by definition, the length of a minimum spanning tree in the subgraph induced by $S \cup \{s\}$.

It is well-known that core(c) is non-empty for MCST-games and core vectors can be found in polynomial time: Suppose T is a minimum spanning tree belonging to a MCST-game. Let x be the allocation vector that allocates to player $i \in N$ the weight of the first edge i encounters on the (unique) path from i to s in T. Granot and Huberman [1981] have proved that $x \in$ core(c).

However, Granot and Huberman [1981] also point out that allocation vectors obtained from the construction above may not be acceptable from a modeling point of view. This motivates the search for allocations for example in the least core and the following generalization of this solution concept. Consider the set of allocation vectors that are optimal solutions of the linear program

$$\begin{array}{lllcll} (P_f) & \max & \epsilon & & & \\ & s.t & x(S) & \leq & c(S) - \epsilon f(S) & \text{for all } S \not\subseteq N, S \neq \emptyset \\ & & x(N) & = & c(N), & \end{array}$$

for a given function $f : 2^N \to \mathbb{R}^+$. Denote this set by f-leastcore(c). Obviously, the larger $f(S)$ is for some coalition $S \subseteq N$, the more decisive S is for determining the optimum value of (P_f). We therefore call a function f as above a *priority function*, which is closely related to the concept of a *taxation function* (see, *e.g.*, Shapley and Shubik [1966], Tijs and Driessen [1986]). Note that $f \equiv 1$ corresponds with the classical least core of Maschler *et al.* [1979]. Moreover, because of the non-emptiness of core(c) of a MCST-game,

$$f\text{-leastcore}(c) \subseteq \text{core}(c) \ \text{ for all } f : 2^N \to \mathbb{R}^+.$$

We prove that the problem of computing an element of f-leastcore(c) of general MCST-games is NP-hard for a large class of priority functions f. This class includes the following examples already known in the literature (see, Faigle and Kern [1993], Shapley and Shubik [1966])

- $f(S) = 1$ for all $S \subsetneq N, S \neq \emptyset$
- $f(S) = c(S)$ for all $S \subsetneq N, S \neq \emptyset$
- $f(S) = |S|$ for all S $\subsetneq N, S \neq \emptyset$.

The proof uses a reduction from minimum cover problems. We show that computing a leastcore-allocation for a special class of graphs introduced in Faigle *et al.* [1997] is already NP-hard. These graphs will be treated in Section 2. Section 3 contains the proof of the theorem. In this section, we also introduce the f-nucleolus which is a generalization of the nucleolus (see Schmeidler [1969]). In Section 4, the functions mentioned above are treated. By giving sufficient conditions for a priority function f to satisfy a number of properties defined in Section 3, we prove that computing an element of f-leastcore(c) of MCST-games is NP-hard for these functions. As a consequence of the main theorem, computing the nucleolus, the nucleon and the per-capita nucleolus of MCST-games is in general NP-hard. We end this section by mentioning some open problems.

2. Exact Cover Graphs

Let $q \in \mathbb{N}$, and let U be a set of $k \geq q$ elements and W be a set of $3q$ elements.

Consider a bipartite graph with node set $U \cup W$ (partitioned into U and W) such that each node $u \in U$ is adjacent to exactly three nodes in W. We say that the node $u \in U$ *covers* its three neighbors in W.

A set $D \subseteq U$ is called a *cover* if each $w \in W$ is incident with some $u \in D$. A *minimum cover* is a cover that minimizes $|D|$. Finding a minimum cover is a well-known NP-hard problem. It includes the NP-complete problem known as EXACT 3-COVER ("X3C") (cf. Garey and Johnson [1979]).

Even finding a minimum cover under the following assumptions is NP-hard.

(**C1**) Each node in W has degree 2 or more.
(**C2**) The size of a minimum cover is at most $q+2$.

This can be shown as follows: Suppose $w \in W$ is a node with degree 1, w is connected to u and u is also connected to w_1 and w_2. Add a vertex $\hat{u}$ to U and connect it to w, w_1 and w_2. The size of a minimum cover will not change. Hence computing the size of a minimum cover, in case (**C1**) holds, is at least as hard as computing the size of a minimimum cover in the general case. To show the validity of (**C2**), add vertices $u_1, u_2, \ldots, u_q$ to U that cover W. Each u_i $(i = 1, \ldots, q)$ covers exactly 3 vertices in W. Next delete u_q. The size of a minimum cover will be less than or equal to $q+2$. If the size is greater than q, the original problem has no exact cover.

If the size of a minimum cover is equal to q, then also delete u_{q-1}. Again the size of a minimum cover will be at most $q+2$. If the size is greater than q, the original problem has no exact cover. If the size is equal to q, also delete u_{q-2} and so on. In each step of the procedure only problems that have a minimum cover with size at most $q+2$ are considered. If u_1 would be deleted, one arrives at the original problem. Hence computing the size of a mimimum cover, in case (**C2**) holds, is at least as hard as computing the size of a minimum cover in the general case.

We construct an MCST-game from a minimum cover problem as follows (cf. Faigle *et al.* [1997]). Define the graph $G = (V, E)$ such that the node set of G consists of $U \cup W$ and three additional nodes: The *Steiner node* St, the *guardian* g, and the *supply* s. The edge set E of G comprises the following:

- all edges e from the bipartite graph on $U \cup W$, each of them having length $l(e) = q + 1$;
- for each $u \in U$, an edge (u, St) between u and St of length $l(u, St) = q$ and an edge (u, g) between u and g of length $l(u, g) = q + 1$;
- an edge (St, g) between St and g of length $l(St, g) = q + 1$;
- an edge (g, s) between g and s of length $l(g, s) = 2q - 1$.

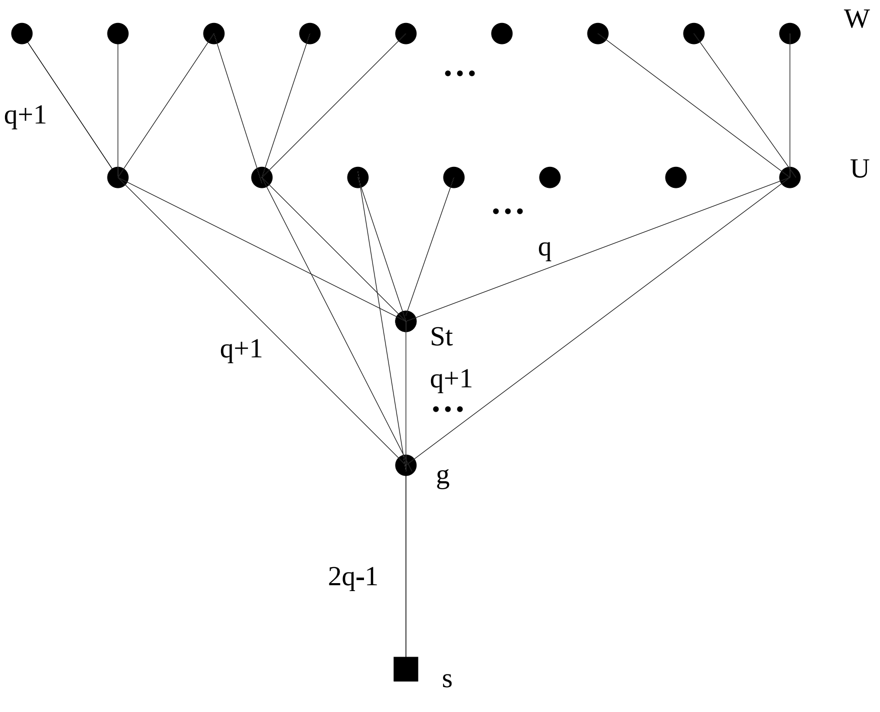

Figure 2.1

We extend G to the complete graph $\overline{G}$ on V with distances induced from G, *i.e.*, if $e=(i,j)$ is an edge in $\overline{G}$, then $l(i,j)$ is the length of a shortest path from i to j in G.

A minimum spanning tree ("MST") in $\overline{G}$ is obtained by connecting each $w \in W$ to some $u \in U$ by which it is covered. Such a $u \in U$ exists because each node $w \in W$ has a neighbor in U (indeed, it has at least 2 neighbors in U). Then one connects each $u \in U$ to St, and finally connects St to g and g to s. The resulting MST has a total length of

$$c(N) \;=\; 3q(q+1)+kq+3q.$$

Furthermore note that, by (**C1**), each $w \in W$ is covered by at least *two* vertices in U. Hence it is straightforward to see that the following property holds for $\overline{G}$:

(**L**) For each $v \in U \cup W$, there exists a MST T in the graph $\overline{G}$ such that v is a leaf of T.

3. The f-Least Core of Minimum Cover Graphs

Consider a graph $G=(V,E)$ and its completion $\overline{G}$ as described in the previous section. The f-leastcore(c), relative to a priority function $f: 2^N \to \mathbb{R}^+$, of the corresponding MCST-game consists of all allocation vectors that are optimal solutions of the linear program

$$\begin{array}{llrcll} (P_f) & \max & & \epsilon & & \\ & s.t. & x(S) & \leq & c(S)-\epsilon f(S) & \text{for all } S \nsubseteq N, S \neq \emptyset \\ & & x(N) & = & c(N), & \end{array}$$

where $N = V \backslash \{s\}$ and $c(S)$ is the length of a MST in $\overline{G}$ connecting S to the supply s.

A basic observation is now the following. If a node $v \in N$ occurs as a leaf in some MST T for $\overline{G}$ and if e is the unique edge in T incident with v, then $T \backslash e$ is a MST for $V \backslash \{v\}$. Thus $c(N \backslash \{v\}) = c(N) - l(e)$, where $l(e)$ is the length of e.

Hence, by property (**L**) of the previous section, the feasibility constraints of (P_f) imply the following inequalities

$$\begin{array}{rcll} x(w) & \geq & q+1+\epsilon f(N\backslash\{w\}) & (w \in W) \\ x(u) & \geq & q+\epsilon f(N\backslash\{u\}) & (u \in U). \end{array}$$

Furthermore, the coalition $S = N\backslash\{g\}$ can be connected to the supply node s at a total cost of $c(N)$. Hence, the feasibility constraints of (P_f) also imply

$$x(g) \;\geq\; \epsilon f(N\backslash\{g\})\,.$$

This motivates the following definition.

For $\epsilon > 0$, let $x^\epsilon \in \mathbb{R}^N$ be the vector defined by

$$\begin{array}{lcll} x^\epsilon(w) & = & q + \epsilon f(N\backslash\{w\}) & \text{for all } w \in W \\ x^\epsilon(u) & = & q + 1 + \epsilon f(N\backslash\{u\}) & \text{for all } u \in U \\ x^\epsilon(g) & = & \epsilon f(N\backslash\{g\}) & \\ x^\epsilon(St) & = & c(N) - x^\epsilon(U \cup W \cup \{g\}). & \end{array}$$

Motivated by the examples mentioned in Section 1, we restrict our attention to priority functions f that depend only on the size and the cost of a coalition, *i.e.*, we consider functions (also denoted by f) of the type $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}^+$, which we always assume to be efficiently computable, and set $f(S) = f(|S|, c(S))$. For technical reasons, we assume that $f(S) > 0$ whenever $|S| > 0$ and $c(S) > 0$.

It is straightforward to check that the following parameters do not depend on the particular representative $w \in W$ or $u \in U$:

$$\begin{array}{lcll} f^w & := & f(N\backslash\{w\}) & (w \in W) \\ f^u & := & f(N\backslash\{u\}) & (u \in U). \end{array}$$

Define for a cover $D \subseteq U$

$$\epsilon^f(D) \quad = \quad \frac{|D| + 2q - 1}{|D|f^u + 3qf^w + f(N\backslash\{g\}) + f(D \cup W \cup \{g\})},$$

and let

$$\epsilon^f = \min\{\epsilon^f(D) | D \subseteq U \text{ covers } W\}.$$

Remark 3.1 Suppose $D \subseteq U$ is a cover and consider the coalition $S = D \cup W \cup \{g\}$. The cost $c(S)$ is easily seen to be $c(S) = 3q(q+1) + |D|(q+1) + 2q - 1$, *i.e.*, $c(S)$ depends only on $|D|$. Hence also $f(S)$ and, therefore $\epsilon^f(D)$ only depend on $|D|$, *i.e.*, satisfies $\epsilon^f(D_1) = \epsilon^f(D_2)$ if $|D_1| = |D_2|$ for all covers $D_1, D_2 \subseteq U$. As a consequence, we can a priori compute all possible values of $\epsilon^f(D)$ for $|D|$ ranging from q to k.

Lemma 3.1. *If ϵ^* is the optimal value of (P_f) then $\epsilon^* \le \epsilon^f$.*

Proof: Let (x, ϵ^*) be an optimal solution of (P_f). As we have seen, the feasibility constraints imply

$$\begin{array}{lcll} x(w) & \ge & q + 1 + \epsilon^* f^w & (w \in W) \\ x(u) & \ge & q + \epsilon^* f^u & (u \in U) \\ x(g) & \ge & \epsilon^* f(N\backslash\{g\}). & \end{array}$$

Suppose $D \subseteq U$ is a cover for which $\epsilon^f(D) = \epsilon^f$. Consider the coalition $S = \{g\} \cup D \cup W$. Then

$$x(S) \ \ge \ \epsilon^* f(N\backslash\{g\}) + |D|q + \epsilon^*|D|f^u + 3q(q+1) + \epsilon^* 3qf^w$$

whereas,

$$c(S) \ = \ 3q(q+1) + |D|(q+1) + 2q - 1.$$

Since $x(S) \leq c(S) - \epsilon^* f(S)$, we get

$$\epsilon^* \leq \frac{|D| + 2q - 1}{|D| f^u + 3q f^w + f(N \backslash \{g\}) + f(D \cup W \cup \{g\})} = \epsilon^f .$$

$\diamondsuit$

We call a priority function $f : 2^N \to \mathbb{R}^+$ *feasible* if f satisfies the following properties (with respect to MCST-games on minimum cover graphs):

(**P1**) ϵ^f is the optimal value of (P_f).

(**P2**) For a cover $D \subseteq U$ of size $q \leq |D| \leq q + 2$, we have

$$\epsilon^f = \epsilon^f(D) \text{ if and only if } D \subseteq U \text{ is a minimum cover.}$$

Our main result can be formulated as follows:

Theorem 3.1. *For the class of feasible priority functions, the problem of computing an allocation vector $x \in f$-leastcore(c) of MCST-games is NP-hard.*

Proof: First we will show that for all $w \in W$

$$x(w) = q + 1 + \epsilon^f f^w.$$

Suppose (x, ϵ^f) is an optimal solution of (P_f). The feasibility constraints imply

$$\begin{array}{rrcll} & x(w) & \geq & q + 1 + \epsilon^f f^w & (w \in W) \\ & x(u) & \geq & q + \epsilon^f f^u & (u \in U) \\ \text{and} & x(g) & \geq & \epsilon^f f(N \backslash \{g\}). & \end{array}$$

Now let $D \subseteq U$ be a cover for which $\epsilon^f = \epsilon^f(D)$. Consider the coalition $S = \{g\} \cup D \cup W$. Then

$$\begin{aligned} & 3q(q + 1 + \epsilon^f f^w) + |D|(q + \epsilon^f f^u) + \epsilon^f f(N \backslash \{g\}) \\ \leq\ & x(S) \\ \leq\ & c(S) - \epsilon^f f(S) \\ =\ & 3q(q + 1 + \epsilon^f f^w) + |D|(q + \epsilon^f f^u) + \epsilon^f f(N \backslash \{g\}). \end{aligned}$$

Hence $x(S) \leq c(S) - \epsilon^f f(S)$ implies that for all $w \in W$

$$x(w) = q + 1 + \epsilon^f f^w.$$

Hence $x \in f$-leastcore(c) provides us with the value of the parameter ϵ^f. We can efficiently compute the size $|D|$ of a minimum cover $D \subseteq U$ as follows: Compute $\epsilon^f(D)$ for $|D| = q$, $|D| = q+1$ and $|D| = q+2$ (cf. Remark 3.1). By (**C2**), it suffices to compute $\epsilon^f(D)$ only for these sizes. By (**P2**), $\epsilon^f = \epsilon^f(D)$ for at least one of these sizes. Note that a cover D of size $|D| \leq k-2$ implies the existence of covers with size $|D|+1$ and $|D|+2$. Hence, by (**P2**), the size of a minimum cover $|D|$ will be the maximum of the sizes for which equality holds.

Given an allocation vector $x \in f$-leastcore(c), we can thus compute the size of a minimum cover D in polynomial time. Hence the computation of such a vector is at least as hard as the computation of the size of a minimum cover.

$\diamondsuit$

Theorem 3.2. *The set of feasible priority functions $f : 2^N \to \mathbb{R}^+$ forms a convex cone (minus $f \equiv 0$).*

Proof: It is obvious that αf is feasible for $\alpha > 0$ if f is feasible. Now suppose $f_1, f_2 : 2^N \to \mathbb{R}^+$ are feasible. We will show that $f := f_1 + f_2$ is feasible. It is straightforward to verify that for all covers $D \subseteq U$

$$\epsilon^f(D) = \frac{\epsilon^{f_1}(D)\epsilon^{f_2}(D)}{\epsilon^{f_1}(D) + \epsilon^{f_2}(D)}.$$

This expression is minimal if and only if $\epsilon^{f_1}(D)$ and $\epsilon^{f_2}(D)$ are minimal. Let $D \subseteq U$ be a cover with $q \le |D| \le q+2$. Since f_1 and f_2 satisfy (**P2**), $\epsilon^{f_1}(D)$ and $\epsilon^{f_2}(D)$ are minimal if and only if D is a minimum cover. Hence f satisfies (**P2**).

To show that (**P1**) holds for f, let $D \subseteq U$ be a minimum cover. Hence $\epsilon^f = \epsilon^f(D)$. We claim that $\epsilon^f = \epsilon^*$, the optimum value of (P_f). By Lemma 3.1, it suffices to show that ϵ^f is a feasible value for (P_f).

Suppose (x^1, ϵ^{f_1}) is an optimal solution for (P_{f_1}) and (x^2, ϵ^{f_2}) is an optimal solution for (P_{f_2}). Define

$$x(v) = \frac{\epsilon^{f_2} x^1(v) + \epsilon^{f_1} x^2(v)}{\epsilon^{f_1} + \epsilon^{f_2}} \quad \text{for all} \quad v \in N.$$

(Note that $x(N) = c(N)$ because x^1 and x^2 are allocations.)

Suppose $S \not\subseteq N, S \neq \emptyset$. Then

$$\begin{aligned} c(S) - \epsilon^f f(S) &= c(S) - \frac{\epsilon^{f_1}\epsilon^{f_2}}{\epsilon^{f_1} + \epsilon^{f_2}}(f_1(S) + f_2(S)) \\ &= \frac{\epsilon^{f_2}(c(S) - \epsilon^{f_1} f_1(S)) + \epsilon^{f_1}(c(S) - \epsilon^{f_2} f_2(S))}{\epsilon^{f_1} + \epsilon^{f_2}} \\ &\ge \frac{\epsilon^{f_2} x^1(S) + \epsilon^{f_1} x^2(S)}{\epsilon^{f_1} + \epsilon^{f_2}} \\ &= x(S). \end{aligned}$$

$\diamondsuit$

Priority functions f suggest to extend the notion of the classical nucleolus tot the f-nucleolus as follows: We define the *excess* of a non-empty coalition $S \subsetneq N$ (with respect to x) as the number

$$e(S,x) \;=\; \begin{cases} \dfrac{c(S)-x(S)}{f(S)} & \text{if } f(S)>0 \\ \infty & \text{if } f(S)=0. \end{cases}$$

The *excess vector* $\Theta(x)$ is obtained by ordering the 2^N-2 excess values $e(S,x)$ in a non-decreasing sequence. The *f-nucleolus* is then defined to be the set of all allocation vectors $x \in \mathbb{R}^N$ that lexicographically maximize the excess vector $\Theta(x)$. If f only depends on the size of a coalition, *i.e.*, $f(S) = f(T)$ if $|S|=|T|$ for all coalitions $S,T \neq \emptyset, N$, f-nucleolus(c) coincides with the f-nucleolus of Wallmeier [1983].

For f given by $f(S)=1$ for all $S \neq \emptyset, N$, the f-nucleolus is equal to the nucleolus (see Schmeidler [1969]). For f given by $f(S)=c(S)$ for all $S \neq \emptyset, N$, the f-nucleolus is called the *nucleon* (see Faigle *et al.* [1998a]) and for f given by $f(S)=|S|$ for all $S \neq \emptyset, N$, the f-nucleolus is called the *per-capita nucleolus* (see, *e.g.*, Young *et al.* [1982]).

Because it is clear that f-nucleolus$(c) \subseteq f$-leastcore(c), the following corollary holds.

Corollary 3.1. *For the class of feasible priority functions, the problem of computing an allocation vector $x \in f$-nucleolus(c) of MCST-games is NP-hard.*

4. Sufficient Conditions

In this section, we assume that a priority function $f : 2^N \to \mathbb{R}^+$ satisfies the following conditions with respect to MCST-games on minimum cover graphs. Thereby we call a coalition $S \subset N$ *connected*, if the induced subgraph $G(S) = (V(S), E(S))$ is connected.

Conditions:

(**S1**) $f^w \;\leq\; f^u \;\leq\; (1+\frac{1}{q})f^w$

(**S2**) There exists a number $M \in \mathbb{R}^+$, independent of q and k, for which

$$f(S) \leq M f^w \quad \text{for all } S \subsetneq N, S \neq \emptyset.$$

(**S3**) For all $S, S' \subsetneq N$ with S, S' connected, $|S| > \frac{1}{3}q$ and $0 \leq |S'|-|S| \leq 2$

$$|f(S') - f(S)| \leq \frac{1}{4} f^w.$$

Theorem 4.1. *Let the priority function $f : 2^N \to \mathbb{R}^+$ satisfy conditions (**S1**), (**S2**) and (**S3**). Then f satisfies (**P1**) and (**P2**), provided q is sufficiently large.*

Proof: Let $D \subseteq U$ be a cover with minimum size. We will prove that $x := x^{\epsilon^f(D)}$ and $\epsilon := \epsilon^f(D)$ are feasible for (P_f). By Lemma 3.1 and the definition of ϵ^f, $\epsilon^f = \epsilon^f(D)$ is then the optimal value of (P_f). Because $\epsilon^f(D)$ and, therefore ϵ^f only depends on $|D|$ (cf. Remark 3.1), D can be any minimum cover. Finally, we will show that $\epsilon^f < \epsilon^f(D)$ for all covers $D \subseteq U$ that are not minimal.

Let $S \not\subseteq N, S \neq \emptyset$ maximize $\delta(S) := x(S) - c(S) + \epsilon f(S)$. We have to show that $\delta(S) \leq 0$. Suppose $\delta(S) > 0$.

Recall that,

$$\begin{array}{lcll} x(w) & = & q+1+\epsilon f^w & \text{for all } w \in W \\ x(u) & = & q+\epsilon f^u & \text{for all } u \in U \\ x(g) & = & \epsilon f(N\backslash\{g\}). & \end{array}$$

For the rest of the proof, we need the following relations.

(4.1) $\dfrac{2}{3}\dfrac{1}{f^w} \leq \epsilon \leq \dfrac{3}{4}\dfrac{1}{f^w}$.

(4.2) If $S \not\subseteq N$ connected and $\delta(S) > 0$, then $|S| > \dfrac{1}{3}q$.

Proof of (4.1): We have

$$\begin{array}{rl} \epsilon \quad = & \dfrac{|D|+2q-1}{|D|f^u+3qf^w+f(N\backslash\{g\})+f(D\cup W\cup\{g\})} \\[2ex] \leq & \dfrac{|D|+2q-1}{|D|f^u+3qf^w} \\[2ex] \leq^{(S1)} & \dfrac{|D|+2q-1}{|D|f^w+3qf^w} \\[2ex] \leq^{(C2)} & \dfrac{3q+1}{4q+2}\dfrac{1}{f^w} \\[2ex] \leq & \dfrac{3}{4}\dfrac{1}{f^w}, \end{array}$$

and

$$\begin{aligned}
\epsilon &= \frac{|D|+2q-1}{|D|f^u+3qf^w+f(N\backslash\{g\})+f(D\cup W\cup\{g\})} \\
&\geq^{(S1),(S2)} \frac{|D|+2q-1}{|D|(1+\frac{1}{q})f^w+3qf^w+2Mf^w} \\
&\geq^{(C2)} \frac{|D|+2q-1}{|D|+3q+2M+2}\frac{1}{f^w} \\
&\geq \frac{3q-1}{4q+2M+2}\frac{1}{f^w} \qquad \text{(since } |D|\geq q\text{)} \\
&\geq \frac{2}{3}\frac{1}{f^w} \qquad \text{(for } q \text{ sufficiently large).}
\end{aligned}$$

Proof of (4.2): First we show that $x(St)\leq \frac{1}{3}q$. We have

$$\begin{aligned}
x(St) &= c(N)-x(g)-kx(u)-3qx(w) \\
&\leq 3q+3q(q+1)+kq-kq-\epsilon kf^u-3q(q+1)-\epsilon 3qf^w \\
&= 3q-\epsilon kf^u-\epsilon 3qf^w \\
&\leq 3q-\epsilon qf^u-\epsilon 3qf^w \qquad \text{(since } k\geq q\text{)} \\
&\leq^{(S1)} 3q-\epsilon 4qf^w \\
&\leq^{(4.1)} \tfrac{1}{3}q.
\end{aligned}$$

Hence, in particular, $x(St)\leq q+1+\epsilon f^u$. Thus, for $S\not\subseteq N$, we have

$$\begin{aligned}
x(S) &\leq^{(S1)} (q+1+\epsilon f^u)|S|+\epsilon f(N\backslash\{g\}) \\
&\leq^{(S1),(S2)} (q+1+\epsilon(1+\tfrac{1}{q})f^w)|S|+\epsilon Mf^w \\
&\leq^{(4.1)} (q+2)|S|+M, \\
c(S) &\geq q|S|+q-1,
\end{aligned}$$

and

$$\begin{aligned}
\epsilon f(S) &\leq^{(S2)} \epsilon Mf^w \\
&\leq^{(4.1)} M.
\end{aligned}$$

Hence

$$\begin{aligned} 0 &< x(S) - c(S) + \epsilon f(S) \\ &\leq (q+2)|S| + M - q|S| - q + 1 + M \\ &= 2|S| - q + 1 + 2M. \end{aligned}$$

Then $|S| > \frac{1}{2}q - \frac{1}{2} - M > \frac{1}{3}q$ (for q sufficiently large).

This completes the proof of (4.2). We now continue the proof of the theorem by establishing a sequence of claims.

Claim (1): If $St \in \mathcal{S}$ then $|S| < |N| - 1$.

Suppose $S = N \backslash \{v\}$ for some $v \in \{g\} \cup U \cup W$, then $\delta(S) = 0$ by definition of x.

Claim (2): $St \in \mathcal{S}$ or $g \in S$.

Suppose $S \subseteq U \cup W, S = S_1 \cup S_2 \cup \ldots \cup S_r$ with S_i $(i = 1, \ldots, r)$ connected. Then

$$\begin{aligned} x(S) &= |S \cap U|\epsilon f^u + |S \cap W|\epsilon f^w + |S \cap U|q + |S \cap W|(q+1) \\ &\leq^{(S1)} |S \cap U|\epsilon(1 + \tfrac{1}{q})f^w + |S \cap W|\epsilon f^w + |S \cap U|q + |S \cap W|(q+1) \\ &\leq^{(4.1)} \tfrac{3}{4}|S \cap U|(1 + \tfrac{1}{q}) + \tfrac{3}{4}|S \cap W| + |S \cap U|q + |S \cap W|(q+1) \\ &= \tfrac{4q^2+3q+3}{4q}|S \cap U| + (q + \tfrac{7}{4})|S \cap W|, \\ c(S) &\geq 3q + 2q(r-1) + \textstyle\sum_{i=1}^r (|S_i| - 1)(q+1) \\ &= r(q-1) + q + |S|(q+1) \end{aligned}$$

and

$$\begin{aligned} \epsilon f(S) &\leq^{(S2)} M\epsilon f^w \\ &\leq^{(4.1)} M. \end{aligned}$$

Note that for $i = 1, \ldots, r$

$$|S_i \cap W| \leq 2|S_i \cap U| + 1.$$

Hence

$$|S \cap U| = \sum_{i=1}^r |S_i \cap U| \geq \sum_{i=1}^r \frac{|S_i \cap W| - 1}{2} = \frac{1}{2}|S \cap W| - \frac{1}{2}r.$$

Then

$$\begin{aligned}
\delta(S) &= x(S) - c(S) + \epsilon f(S) \\
&\leq \tfrac{4q^2+3q+3}{4q}|S \cap U| + (q + \tfrac{7}{4})|S \cap W| - r(q-1) - q - |S|(q+1) + M \\
&= \tfrac{3}{4}|S \cap W| - \tfrac{q-3}{4q}|S \cap U| - r(q-1) - q + M \\
&\leq \tfrac{3}{4}|S \cap W| + \tfrac{q-3}{4q}(\tfrac{1}{2}r - \tfrac{1}{2}|S \cap W|) - r(q-1) - q + M \\
&\leq M + 2 - \tfrac{1}{8}q \qquad \text{(since } |S \cap W| \leq 3q \text{ and } r \geq 1) \\
&\leq 0 \qquad \text{(for } q \text{ sufficiently large).}
\end{aligned}$$

Hence $g \in S$ or $St \in S$.

Claim (3): $S \cap U$ covers $S \cap W$.

Up to now, we have proved that a MST for S looks as follows. Each $u \in S \cap U$ is connected to g (with cost $q+1$) or to St (with cost q). Each covered $w \in S \cap W$ is connected to a vertex $u \in S \cap U$ (with cost $q+1$). Each uncovered $w \in S \cap W$ is w.l.o.g. joined to g (with cost $2q+2$) or to St (with cost $2q+1$). Now suppose $w \in S \cap W$ is not covered by $S \cap U$. Suppose w is covered by $u (\notin S)$. Then

$$c(S \backslash \{w\} \cup \{u\}) \leq c(S) - (q+1),$$

and

$$\begin{aligned}
& \delta(S \backslash \{w\} \cup \{u\}) - \delta(S) \\
= \quad & x(u) - x(w) + c(S) - c(S \backslash \{w\} \cup \{u\}) + \epsilon(f(S \backslash \{w\} \cup \{u\}) - f(S)) \\
\geq^{(S2)} \quad & \epsilon(f^u - f^w) + q - \epsilon M f^w \\
\geq^{(S1),(4.1)} \quad & q - \tfrac{3}{4}M \\
> \quad & 0 \qquad \text{(for } q \text{ sufficiently large),}
\end{aligned}$$

contradicting the maximality of $\delta(S)$. Hence $S \cap U$ covers $S \cap W$. In particular, S is connected and, by (4.2), $|S| > \frac{1}{3}q$.

Claim (4): S contains all w covered by $S \cap U$.

Suppose $w \in W$ is covered by $S \cap U$ and $w \notin S$. By (1), $|N \backslash S| \geq 2$. Then

$$\begin{aligned}
\delta(S \cup \{w\}) - \delta(S) &= x(w) + c(S) - c(S \cup \{w\}) + \epsilon(f(S \cup \{w\}) - f(S)) \\
&= \epsilon(f^w + f(S \cup \{w\}) - f(S)) \\
&>^{(S3)} 0,
\end{aligned}$$

contradicting the maximality of $\delta(S)$.

Claim (5): $St \notin S$.

Suppose $St \in S$. If $S \cap U = U$ then, by (4), $S \cap W = W$. Hence $S = N \backslash \{g\}$ in contradiction to (1). Suppose $u \notin S$. By (1), $|N \backslash S| > 1$. Then

$$\begin{aligned}
\delta(S \cup \{u\}) - \delta(S) &= x(u) + c(S) - c(S \cup \{u\}) + \epsilon(f(S \cup \{u\}) - f(S)) \\
&= \epsilon(f^u + f(S \cup \{u\}) - f(S)) \\
&\geq^{(S1)} \epsilon(f^w + f(S \cup \{u\}) - f(S)) \\
&>^{(S3)} 0,
\end{aligned}$$

contradicting the maximality of $\delta(S)$.

Claim (6): $S \cap W = W$.

Suppose $w \in W \backslash S$. By (4), w is not covered by $S \cap U$. Because each vertex in W has at least two neighbors in U, we have $|S \cap U| \leq |U| - 2$. Suppose w is covered by $u (\notin S)$. Then

$$\begin{aligned}
& \delta(S \cup \{u\} \cup \{w\}) - \delta(S) \\
= \;& x(u) + x(w) + c(S) - c(S \cup \{u\} \cup \{w\}) + \epsilon(f(S \cup \{u\} \cup \{w\}) - f(S)) \\
= \;& -1 + \epsilon(f^u + f^w + f(S \cup \{u\} \cup \{w\}) - f(S)) \\
>^{(S1),(S3)} \;& -1 + \epsilon \tfrac{3}{2} f^w \\
\geq^{(4.1)} \;& 0,
\end{aligned}$$

contradicting the maximality of $\delta(S)$.

Claim (7): $S = \{g\} \cup D \cup W$ for some minimum cover $D \subseteq U$.

Up to now, we have proved that $S = \{g\} \cup D' \cup W$ for some cover $D' \subseteq U$. Suppose $\overline{D} \subseteq U$ is a cover with $|\overline{D}| < |D'|$. W.l.o.g. we may assume that $|\overline{D}| = |D'| - 1$ (otherwise add a sufficient number of vertices $u \in U$ to $\overline{D}$). Let $\overline{S} = \{g\} \cup \overline{D} \cup W$. It is obvious that $x(\overline{S}) = x(S) - q - \epsilon f^u$ and $c(\overline{S}) = c(S) - q - 1$. Then

$$\begin{aligned}
& \delta(\overline{S}) - \delta(S) \\
= \;& 1 + \epsilon(f(\overline{S}) - f(S) - f^u) \\
>^{(S1),(S3)} \;& 1 - \epsilon \tfrac{4}{3} f^w \\
\geq^{(4.1)} \;& 0,
\end{aligned}$$

contradicting the maximality of $\delta(S)$.

We have proved that

$$S = \{g\} \cup D \cup W \text{ for some minimum cover } D \subseteq U,$$

Then (by definition of ϵ) $\delta(S) = 0$. Hence (x, ϵ) is a feasible solution for (P_f), which we had to show.

We complete the proof by showing that $\epsilon^f < \epsilon^f(D)$ for all covers $D \subseteq U$ that are not minimal. Let $D' \subseteq U$ be a cover that is not minimal. We have already shown that $\delta(S) = x^{\epsilon^f}(S) - c(S) + \epsilon^f f(S) = 0$ for all coalitions $S = \{g\} \cup D \cup W$ where $D \subseteq U$ is a minimum cover and that $\delta(S) \geq \delta(T)$ for all coalitions $T \neq \emptyset, N$. Furthermore, from Claim (7), we know that there exists a cover $\overline{D} \subseteq U$ with $|\overline{D}| = |D'| - 1$ and $\delta(\{g\} \cup \overline{D} \cup W) > \delta(\{g\} \cup D' \cup W)$. Then we have $\delta(\{g\} \cup D' \cup W) < 0$, which is equivalent to $\epsilon^f < \epsilon^f(D')$.

$\diamondsuit$

It is straightforward to see that the priority functions f given by

- $f(S) = 1$ for all $S \not\subseteq N, S \neq \emptyset$
- $f(S) = c(S)$ for all $S \not\subseteq N, S \neq \emptyset$
- $f(S) = |S|$ for all S $\not\subseteq N, S \neq \emptyset$

satisfy (**S1**), (**S2**) and (**S3**). Hence, by Theorem 3.1 and Theorem 4.1, the problem of computing an allocation vector $x \in f$-leastcore(c) of MCST-games is NP-hard for these functions. By Corollary 3.1, the problem of computing the nucleolus, the nucleon and the per-capita nucleolus of MCST-games is also NP-hard. Furthermore, one can verify that functions such as, *e.g.*,

- $f(S) = c(S)|S|$ for all $S \not\subseteq N, S \neq \emptyset$
- $f(S) = \dfrac{c(S)}{|S|}$ for all $S \not\subseteq N, S \neq \emptyset$

satisfy (**S1**), (**S2**) and (**S3**). Hence these functions also belong to the class of feasible priority functions.

We end our discussion by mentioning some priority functions for which our approach does not yield any NP-hardness result. In particular, functions that give high priority to small conditions, such as, *e.g.*,

- $f(S) = e^{-|S|}$ for all $S \not\subseteq N, S \neq \emptyset$
- $f(S) = \dfrac{1}{|S|}$ for all $S \not\subseteq N, S \neq \emptyset$

violate conditions (**S2**) and (**S3**).

As an extreme case, suppose there is a set $T \subseteq N$ of important individuals. One may then consider the priority function

$$f(S) \quad = \quad \begin{cases} 1 & \text{if } S = \{i\},\ i \in T \\ 0 & \text{else.} \end{cases}$$

We do not know whether the f-least core or f-nucleolus for any of these functions can be computed efficiently.

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