Semidefinite Programming
Lecture 9

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Applications of SDP in Combinatorial Optimization

Stable Sets in Graphs and Orthonormal Representations

1 Notations

This lecture is based mostly on Chapter 9 (in particular on Section 9.3) of the book of Grötschel, Lovász, Schrijver: Geometric Algorithms and Combinatorial Optimization and we are going to use also the exact same notation was used by them.

Let $G = (V, E)$ be a graph, Q denote the set of cliques of G, and let $\mathbf{w} \in R^{|V|}$. Let S_N be the unit sphere in R^N, i.e. $S_N = \{\mathbf{x} \in R^N \;:\; \|\mathbf{x}\|_2 = 1\}$. Let $ONR(G, N)$ be the set of orthonormal representations of G in R^N, i.e.

$$ONR(G,N) = \{(\mathbf{v}_i \mid i \in V) \;:\; \mathbf{v}_i \in R^N, \mathbf{v}_i \in S_N \;\; \forall i \in V \text{ and } \mathbf{v}_i^T \mathbf{v}_j = 0 \;\; \forall \{i,j\} \notin E\}.$$

Last time, we introduced the following notations:

$$STAB(G) = Conv\{\mathbf{x} \in R^{|V|} \;:\; \mathbf{x} \text{ is the characteristic vector of some stable set of } G\}$$

$$QSTAB(G) = \{\mathbf{x} \in R^{|V|} \;:\; \mathbf{x} \geq 0,\; 1_K^T \mathbf{x} \leq 1 \;\forall K \in Q\}$$

$$TH(G) = \{\mathbf{x} \in R^{|V|} \;:\; \mathbf{x} \geq 0,\;\; \textstyle\sum_{i \in V} (\mathbf{c}^T \mathbf{v}_i)^2 x_i \leq 1 \;\forall \mathbf{c} \in S_N \;\forall (\mathbf{v}_i \mid i \in V) \in ONR(G,N)\}$$

$$\alpha(G, w) = \max\{\mathbf{w}^T \mathbf{x} :\; \mathbf{x} \in STAB(G)\}$$

$$\vartheta(G, w) = \max\{\mathbf{w}^T \mathbf{x} :\; \mathbf{x} \in TH(G)\}$$

$$\alpha^*(G, w) = \max\{\mathbf{w}^T \mathbf{x} :\; \mathbf{x} \in QSTAB(G)\}$$

We know from some of the simple observations were made last time that:

$$STAB(G) \subseteq TH(G) \subseteq QSTAB(G)$$

$$\alpha(G, \mathbf{w}) \le \vartheta(G, \mathbf{w}) \le \alpha^*(G, \mathbf{w})$$

In general $TH(G)$ is not a polytope. In fact, we will prove later (and that is our final goal in this application of SDP) that $TH(G)$ is a polytope if and only if G is perfect. To do this, we need to introduce some further values and prove the Theorem below.

2 Equivalent Formulations for $\vartheta(G, \mathbf{w})$

Let $(\mathbf{v}_i \mid i \in V) \in ONR(G, N)$ and $\mathbf{c} \in S_N$. Let

$$\mathcal{S} = \{A \in R^{|V| \times |V|} \mid A \text{ symmetric}\}$$

$$\mathcal{M} = \{A = (a_{ij}) \in \mathcal{S} \mid a_{ij} = 0 \ \forall \{i, j\} \in E\}$$

$$\mathcal{M}^{\perp} = \{A = (a_{ij}) \in \mathcal{S} \mid a_{ii} = 0 \ \forall i \in V, \ a_{ij} = 0 \ \forall \{i, j\} \notin E\}$$

$$\overline{\mathbf{w}} = (\sqrt{w_i} \mid i \in V) \in R^{|V|}$$

$$W = \overline{\mathbf{w}}\overline{\mathbf{w}}^T = (\sqrt{w_i w_j} \mid i, j \in V) \in \mathcal{S}$$

$$\vartheta_1(G, \mathbf{w}) = \min\{\ \max\{\ \tfrac{w_i}{(\mathbf{c}^T \mathbf{v}_i)^2} \ : \ i \in V\} \ : \ \mathbf{c} \in S_N, \ (\mathbf{v}_i \mid i \in V) \in ONR(G, N)\}$$

$$\vartheta_2(G, \mathbf{w}) = \min\{\lambda_1(X + W) \ : \ X \in \mathcal{M}^{\perp}\}$$

$$\vartheta_3(G, w) = \max\{W \bullet Y \ : \ \mathrm{tr}(Y) = 1, \ Y \in \mathcal{M}, Y \succeq 0\}$$

$$\vartheta_4(G, \mathbf{w}) = \max\{\textstyle\sum_{i \in V} (\mathbf{d}^T \mathbf{z}_i)^2 \ : \ \mathbf{d} \in S_N, \ (\mathbf{z}_i \mid i \in V) \in ONR(\overline{G}, N)\}$$

THEOREM 2.1. $\vartheta(G, \mathbf{w}) = \vartheta_1(G, \mathbf{w}) = \vartheta_2(G, \mathbf{w}) = \vartheta_3(G, \mathbf{w}) = \vartheta_4(G, \mathbf{w})$.

Proof. We will show that:

$$\vartheta(G, \mathbf{w}) \le \vartheta_1(G, \mathbf{w}) \le \vartheta_2(G, \mathbf{w}) \le \vartheta_3(G, \mathbf{w}) \le \vartheta_4(G, \mathbf{w}) \le \vartheta(G, \mathbf{w}).$$

It is easy to see that the above statement is true if $\mathbf{w} = 0$. So we will assume in the following that $\mathbf{w} \neq 0$.

Part 1: Let $\mathbf{x} \in TH(G)$ be a vector such that $\mathbf{w}^T\mathbf{x} = \vartheta(G, \mathbf{w})$, let $(\mathbf{v}_i \mid i \in V) \in ONR(G, N)$ and $\mathbf{c} \in S_N$. Then:

$$\begin{aligned} \mathbf{w}^T\mathbf{x} &= \vartheta(G,\mathbf{w}) = \textstyle\sum_{i\in V} w_i x_i = \sum_{i\in V} \frac{w_i(\mathbf{c}^T\mathbf{v}_i)^2}{(\mathbf{c}^T\mathbf{v}_i)^2} x_i \\ &\le (\,\max_i \tfrac{w_i}{(\mathbf{c}^T\mathbf{v}_i)^2}) \textstyle\sum_{i\in V} (\mathbf{c}^T\mathbf{v}_i)^2 x_i \le \max_i \frac{w_i}{(\mathbf{c}^T\mathbf{v}_i)^2} \end{aligned}$$

Since this is true for any orthonormal representation $(\mathbf{v}_i \mid i \in V) \in ONR(G, N)$ and for any $\mathbf{c} \in S_N$, we have $\vartheta(G, \mathbf{w}) \le \vartheta_1(G, \mathbf{w})$.

Part 2: Let $X \in \mathcal{M}^\perp$ and $t = \lambda_1(X+W)$. Since $tI - X - W \succeq 0$, there is a matrix $Z \in R^{|V|\times|V|}$ such that $Z^TZ = tI - X - W$. Let z_i be the i^{th} column of Z. Then we have:

$$z_i^T z_i = t - w_i \quad \text{for all} \quad i \in V$$

$$z_i^T z^j = -\sqrt{w_i w_j} \quad \text{for all} \quad \{i,j\} \notin E$$

Since Z is singular, there is a $\mathbf{c} \in R^{|V|}$ such that $\mathbf{c} \perp \text{span}\{\mathbf{z}_i \ : \ i \in V\}$. Let

$$v_i = \sqrt{\frac{w_i}{t}}\mathbf{c} + \frac{1}{\sqrt{t}} z_i$$

Then we obtain:

$$v_i^T v_j = \frac{\sqrt{w_i w_j}}{t}\mathbf{c}^T\mathbf{c} + \frac{-\sqrt{w_i w_j}}{t} = 0 \quad \text{for all} \quad \{i,j\} \notin E$$

$$v_i^T v_i = \frac{w_i}{t} + \frac{t - w_i}{t} = 1 \quad \text{for all} \quad i \in V$$

Hence, we have an orthonormal representation of G, and so:

$$\vartheta_1(G,\mathbf{w}) \le \max_i \frac{w_i}{(\mathbf{c}^T\mathbf{v}_i)^2} = \max_i \frac{w_i}{w_i/t} = t = \lambda_1(X+W) = \vartheta_2(G,\mathbf{w}).$$

Part 3: Now, here is the point where we can use semidefinite programming, since the $\vartheta_2(G, \mathbf{w}) \le \vartheta_3(G, \mathbf{w})$ relation follows simply from the duality theory of SDP.

Part 4: Let $Y \in \mathcal{M}$ be such that $Y \succeq 0$, $\text{tr}(Y) = 1$, and $\vartheta_3(G, \mathbf{w}) = W \bullet Y$. Since $Y \succeq 0$, we can write it in the form $Y = U^TU$, for some $U \in R^{|V|\times|V|}$. Let $\mathbf{u}_i$ be the i^{th} column of U, and let $\mathbf{v}$ be a fixed vector from an orthonormal basis of $(\text{lin}\{\mathbf{u}_i \mid \mathbf{u}_i \neq 0\})^\perp$. Then let for each $i \in V$:

$$\mathbf{v}_i = \begin{cases} \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|} & \text{if } \mathbf{u}_i \neq 0, \\ \mathbf{v} & \text{if } \mathbf{u}_i = 0. \end{cases}$$

By using this definition of $\mathbf{v}_i$ and the fact that $U^TU \in \mathcal{M}$, it is easy to see that:

$$\mathbf{v}_i^T \mathbf{v}_j = \begin{cases} 0 & \text{if } \{i,j\} \in E, \\ 1 & \text{if } i = j, \end{cases}$$

which means, that $(\mathbf{v}_i \mid i \in V) \in ONR(\overline{G}, \mathbf{w})$. Since $(U\overline{\mathbf{w}})^T(U\overline{\mathbf{w}}) = \overline{\mathbf{w}}^T U^T U \overline{\mathbf{w}} = \overline{\mathbf{w}}^T Y \overline{\mathbf{w}} = \vartheta_3$, the vector $\mathbf{d} = \frac{1}{\sqrt{\vartheta_3}} U\overline{\mathbf{w}}$ is a vector of unit length. Moreover, for $\mathbf{u}_i \neq 0$, we have:

$$\mathbf{d}^T \mathbf{v}_i = \frac{\overline{\mathbf{w}}^T U^T U e_i}{\sqrt{\vartheta_3}\|\mathbf{u}_i\|},$$

and so the following holds for all $i \in V$:

$$\|\mathbf{u}_i\| \mathbf{d}^T \mathbf{v}_i = \frac{1}{\sqrt{\vartheta_3}} \overline{\mathbf{w}}^T Y e_i.$$

Hence:

$$\sum_{i \in V} \|\mathbf{u}_i\| \sqrt{w_i} \mathbf{d}^T \mathbf{v}_i = \frac{1}{\sqrt{\vartheta_3}} \sum_{i \in V} \overline{\mathbf{w}}^T Y e_i \sqrt{w_i} = \frac{1}{\sqrt{\vartheta_3}} \overline{\mathbf{w}}^T Y \overline{\mathbf{w}} = \sqrt{\vartheta_3}.$$

By the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \vartheta_3 &= (\textstyle\sum_{i \in V} \|\mathbf{u}_i\| \sqrt{w_i} \mathbf{d}^T \mathbf{v}_i)^2 \\ &\leq (\textstyle\sum_{i \in V} \|\mathbf{u}_i\|^2)(\sum_{i \in V} w_i (\mathbf{d}^T \mathbf{v}_i)^2) \\ &= (\mathrm{tr}(Y))(\textstyle\sum_{i \in V} w_i (\mathbf{d}^T \mathbf{v}_i)^2) \\ &= \textstyle\sum_{i \in V} w_i (\mathbf{d}^T \mathbf{v}_i)^2 \\ &\leq \vartheta_4(G, \mathbf{w}). \end{aligned}$$

Part 5: Let $(\mathbf{v}_i \mid i \in V) \in ONR(\overline{G}, N)$ and $\mathbf{d} \in S_N$ such that $\vartheta_4(G, \mathbf{w}) = \sum_{i \in V} (\mathbf{d}^T \mathbf{v}_i)^2 w_i$.

We will show that $((\mathbf{d}^T \mathbf{v}_i)^2 \mid i \in V)^T \in TH(G)$,which will prove also, by the definition of $\vartheta(G, \mathbf{w})$, the relation of $\vartheta_4(G, \mathbf{w}) \leq \vartheta(G, \mathbf{w})$. To do this, let $(\mathbf{u}_i \mid i \in V) \in ONR(G, N)$ and $\mathbf{c} \in S_N$ be arbitrary. We consider the matrices $\mathbf{u}_i \mathbf{v}_i^T$. It is easy to see that:

$$(\mathbf{u}_i \mathbf{v}_i^T) \bullet (\mathbf{u}_j \mathbf{v}_j^T) = (\mathbf{u}_i^T \mathbf{u}_j)(\mathbf{v}_i^T \mathbf{v}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the matrix $\mathbf{c}\mathbf{d}^T$ has unit length with respect to this inner product, since:

$$\|\mathbf{c}\mathbf{d}^T\|_F = \sqrt{(\mathbf{c}\mathbf{d}^T) \bullet (\mathbf{c}\mathbf{d}^T)} = \sqrt{(\mathbf{c}^T \mathbf{c})(\mathbf{d}^T \mathbf{d})} = 1.$$

Hence: $1 = (\mathbf{c}\mathbf{d}^T) \bullet (\mathbf{c}\mathbf{d}^T) \geq \sum_{i \in V} [(\mathbf{c}\mathbf{d}^T) \bullet (\mathbf{u}_i \mathbf{v}_i^T)]^2 = \sum_{i \in V} (\mathbf{c}^T \mathbf{u}_i)^2 (\mathbf{d}^T \mathbf{v}_i)^2$, i.e. $((\mathbf{d}^T \mathbf{v}_i)^2 \mid i \in V) \in TH(G)$.

■

3 Some Corollaries

COROLLARY 3.0.1. *The fact that $\vartheta_1(G, \mathbf{w}) = \vartheta_2(G, \mathbf{w})$ implies the following: for any graph G and for any weighting $w : V \to R_+$ there exists an orthonormal representation $(\mathbf{u}_i \mid i \in V) \in ONR(G, N)$ and a vector $\mathbf{c} \in S_N$ such that $\vartheta(G, \mathbf{w})(\mathbf{c}^T\mathbf{u}_i)^2 = w_i$ for all $i \in V$.*

COROLLARY 3.0.2. *Let $(\mathbf{v}_i \mid i \in V) \in ONR(\overline{G}, N)$ and let $\mathbf{d} \in S_N$. Let $w : V \to R_+$ be any weighting of the nodes of G, and assume that*

$$\sum_{i \in V} (\mathbf{d}^T\mathbf{v}_i)^2 w_i = \vartheta(G, \mathbf{w}).$$

Then

$$\sum_{i \in V} w_i(\mathbf{d}^T\mathbf{v}_i)\mathbf{v}_i = \vartheta(G, \mathbf{w})\mathbf{d}.$$

Proof. We know from the above theorem and the definition of $\vartheta_4(G, \mathbf{w})$ that for any $\mathbf{y} \in S_N$:

$$\sum_{i \in V} w_i(\mathbf{y}^T\mathbf{v}_i)^2 \leq \vartheta(G, \mathbf{w})$$

and by our assumption the left hand side is maximized by $\mathbf{y} = \mathbf{d}$. On the other hand, the left hand side can be written in the following form too:

$$\sum_{i \in V} w_i(\mathbf{y}^T\mathbf{v}_i)^2 = \mathbf{y}^T(\sum_{i \in V} w_i\mathbf{v}_i\mathbf{v}_i T)\mathbf{y},$$

and so the maximum over all vectors $\mathbf{y}$ of unit length is attained when $\mathbf{y}$ is an eigenvector of $\sum_{i \in V} w_i\mathbf{v}_i\mathbf{v}_i^T$, and the maximum value is the corresponding eigenvalue. Hence $\mathbf{d}$ is an eigenvector of this matrix with the eigenvalue $\vartheta(G, \mathbf{w})$. Thus:

$$\sum_{i \in V} w_i(\mathbf{d}^T\mathbf{v}_i)\mathbf{v}_i = (\sum_{i \in V} w_i\mathbf{v}_i\mathbf{v}_i T)\mathbf{d} = \vartheta(G, \mathbf{w})\mathbf{d}.$$

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By using the above theorem, we can get some other (equivalent) descriptions for the convex set $TH(G)$.

COROLLARY 3.0.3.

(*a*) $TH(G) = \{\mathbf{x} \in R^n_+ \ : \ \vartheta(\overline{G}, \mathbf{x}) \leq 1\}$

(*b*) $TH(G) = \{\mathbf{x} \in R^n_+ \ : \ \mathbf{x}^T\mathbf{y} \leq \vartheta(G, \mathbf{y}) \ \ \forall \mathbf{y} \in R^n_+\}$

(*c*) $TH(G) = \{((\mathbf{d}^T\mathbf{v}_i)^2 \mid i \in V) \ : \ (\mathbf{v}_i \mid i \in V) \in ONR(\overline{G}, N) \ \ \textit{and} \ \ \mathbf{d} \in S_N\}$

Proof.

(a) Since $\vartheta(\overline{G},\mathbf{x}) = \max\{\sum_{i\in V}(\mathbf{c}^T\mathbf{u}_i)^2 x_i \mid (\mathbf{u}_i \mid i\in V)\in ONR(G,N),\ \mathbf{c}\in S_N\}$ and $\vartheta(G,\mathbf{w}) = \vartheta_4(G,\mathbf{w})$, we have the following:

$$\begin{aligned}
&\{\ \mathbf{x}\in R^n_+ \ :\ \vartheta(\overline{G},\mathbf{x})\le 1\}\\
&= \{\mathbf{x}\in R^n_+ \ :\ \max\{\textstyle\sum_{i\in V}(\mathbf{c}^T\mathbf{u}_i)^2 x_i \mid (\mathbf{u}_i\mid i\in V)\in ONR(G,N),\ \mathbf{c}\in S_N\}\le 1\}\\
&= \{\mathbf{x}\in R^n_+ \ :\ \textstyle\sum_{i\in V}(\mathbf{c}^T\mathbf{u}_i)^2 x_i \le 1\ \ \forall(\mathbf{u}_i\mid i\in V)\in ONR(G,N),\ \ \forall\mathbf{c}\in S_N\le 1\}\\
&= TH(G).
\end{aligned}$$

(b) Let $TH'(G)$ denote the set $\{\mathbf{x}\in R^n_+ \ :\ \mathbf{x}^T\mathbf{y}\le\vartheta(G,\mathbf{y})\ \ \forall\mathbf{y}\in R^n_+\}$. We will show that $TH(G) = TH'(G)$. By the definition of $\vartheta(G,\mathbf{y})$, each $\mathbf{x}\in TH(G)$ satisfies $\mathbf{x}^T\mathbf{y}\le\vartheta(G,\mathbf{y})$, so $TH(G)\subset TH'(G)$. To show that $TH'(G)\subset TH(G)$, let $\mathbf{x}\in TH'(G)$. Then $\mathbf{x}^T\mathbf{y}\le\vartheta(G,\mathbf{y})\ \forall\mathbf{y}\ge 0$, in particular for $\mathbf{y} = ((\mathbf{c}^T\mathbf{u}_i)^2 \ :\ i\in V)$ for any $(\mathbf{u}_i\mid i\in V)\in ONR(G,N)$ and for any $\mathbf{c}\in S_N$. For any of these $\mathbf{y}$'s, we have:

$$\begin{aligned}
\vartheta(G,\mathbf{y}) &= \min\{\max\{\tfrac{y_i}{(\mathbf{d}^T\mathbf{v}_i)^2}\ :\ i\in V\}\ (\mathbf{u}_i\ :\ i\in V)\in ONR(G,N),\ \ \mathbf{d}\in S_N\}\\
&= \min\{\max\{\tfrac{(\mathbf{c}^T\mathbf{u}_i)^2}{(\mathbf{d}^T\mathbf{v}_i)^2}\ :\ i\in V\}\ (\mathbf{u}_i\ :\ i\in V)\in ONR(G,N),\ \ \mathbf{d}\in S_N\}\\
&\le \max\{\tfrac{(\mathbf{c}^T\mathbf{u}_i)^2}{(\mathbf{c}^T\mathbf{u}_i)^2}\ :\ i\in V\} = 1.
\end{aligned}$$

So for any $(\mathbf{u}_i\mid i\in V)\in ONR(G,N)$ and for any $\mathbf{c}\in S_N$, we have:

$$\sum_{i\in V} x_i(\mathbf{c}^T\mathbf{u}_i)^2 \le \vartheta(G,((\mathbf{c}^T\mathbf{u}_i)\ :\ i\in V))\le 1,\quad \text{i.e. } \mathbf{x}\in TH(G).$$

(c) Let $TH''(G)$ denote the set $\{((\mathbf{d}^T\mathbf{v}_i)^2\mid i\in V)\ :\ (\mathbf{v}_i\mid i\in V)\in ONR(\overline{G},N)$ and $\mathbf{d}\in S_N\}$. We will show that $TH(G) = TH''(G)$. By proving the inequality of $\vartheta_4\le\vartheta$ in the proof of the theorem above, we have already shown that $((\mathbf{d}^T\mathbf{v}_i)^2\mid i\in V)\in TH(G)\ \forall((\mathbf{v}_i\ :\ i\in V)\in ONR(G,N)$ and $\forall\mathbf{d}\in S_N$, i.e. $TH''(G)\subset TH(G)$. Now, we will show that $TH(G)\subset TH''(G)$. Let $\mathbf{x}\in TH(G)$ and let $t = \vartheta(\overline{G},\mathbf{x})$. Then by part (a) of this corollary, we know that $t\le 1$. By the previous corollary we made, there exists an orthonormal representation $(\mathbf{v}_i\mid i\in V)\in ONR(\overline{G},N)$ and a vector $\mathbf{d}\in S_N$ such that $t(\mathbf{d}^T\mathbf{v}_i)^2 = x_i$ for all $i\in V$. Now, let $\mathbf{d}_1$ be a unit vector orthogonal to both $\mathbf{d}$ and all $\mathbf{v}_i$, $i\in V$ (in any linear space containing R^N). Let $\mathbf{d}_0 = \sqrt{t}\mathbf{d}+\sqrt{1-t}\mathbf{d}_1$. Then $\|\mathbf{d}_0\| = 1$ and $(\mathbf{d}_0^T\mathbf{v}_i)^2 = t(\mathbf{d}^T\mathbf{v}_i)^2 = x_i\ \ \forall i\in V$, which proves that $\mathbf{x}\in TH''(G)$.

■

4 Characterization of Facets of $TH(G)$

DEFINITION 4.1. *(Convex Corner)* $C\subset R^n_+$ *is a* **convex corner** *if* $\mathbf{x}\in C, \mathbf{x}'\ge 0$ *and* $\mathbf{x}'\le\mathbf{x}$ *imply* $\mathbf{x}'\in C$.

DEFINITION 4.2. *(Antiblocker) For any set $S \subset R^n_+$, its* **antiblocker** *is*

$$abl(S) = \{\mathbf{y} \in R^n_+ \mid \mathbf{y}^T\mathbf{x} \leq 1 \quad \forall x \in S\}.$$

We will use the following results without proving them:

THEOREM 4.1. *If A and B are convex corners, then $A \subset B$ implies $abl(B) \subset abl(A)$.*

THEOREM 4.2. *For a convex corner A, $\quad abl(abl(A)) = A$.*

THEOREM 4.3. *If the optimization problem over a convex corner C can be solved in polynomial time, then the optimization problem over $abl(C)$ can be solved also in polynomial time.*

Observations:

(1) abl$(QSTAB(G)) = CLIQUE(G)$

(2) abl$(QCLIQUE(G)) = STAB(G)$

(3) The optimization problem over $QSTAB(G)$ is NP-hard, since it is equivalent with the optimization problem over $CLIQUE(G)$.

(4) By using the previous results, we have the following interesting fact: however $TH(G)$ is sandviched between $STAB(G)$ and $QSTAB(G)$, and the optimization problem is NP-hard over both $STAB(G)$ and $QSTAB(G)$, it can be solved over $TH(G)$ in polynomial time.

THEOREM 4.4. $\quad abl(TH(G)) = TH(\overline{G})$.

Proof. The statement follows from the definitions and from part (c) of Corollary 3.0.3 ■

We have already mentioned that $TH(G)$ is not a polytope in general. We will now characterize its facets.

THEOREM 4.5. *If an inequality defines a facet of $TH(G)$, then it is a positive multiple of one of the nonnegativity constraints or of one of the clique constraints.*

Proof. Suppose that $\mathbf{a}^T\mathbf{x} \leq \alpha$ defines a facet of $TH(G)$. Let $F = \{\mathbf{x} \in R^{|V|} : \mathbf{a}^T\mathbf{x} = \alpha\} \cap TH(G)$ and let $\mathbf{z}$ be a point in the relative interior of F. Then either $z_i = 0$ for some i, or by part (a) of Corollary 3.0.3 $\vartheta(\overline{G}, \mathbf{z}) = 1$. In the first case $\mathbf{a}^T\mathbf{x} \leq \alpha$ is obviously equivalent to $x_i \geq 0$. So let us assume in the following that $\vartheta(\overline{G}, \mathbf{z}) = 1$. Since $\vartheta = \vartheta_4$, there exists a $(\mathbf{u}_i \ : \ i \in V) \in ONR(G, N)$ and $\mathbf{c} \in S_N$ such that:

$$\sum_{i \in V} z_i (\mathbf{c}^T \mathbf{u}_i)^2 = 1.$$

On the other hand, all $\mathbf{x} \in TH(G)$ satisfy the folowing constraint:

$$\sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i)^2 \leq 1.$$

Thus, it follows that $\sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i)^2 \leq 1$ must be equivalent with $\mathbf{a}^T \mathbf{x} \leq \alpha$, so we may assume that $\alpha = 1$ and $a_i = (\mathbf{c}^T \mathbf{u}_i)^2$. Since the equation

$$\sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i)^2 = 1 = \vartheta(\overline{G}, \mathbf{x}).$$

holds for all $\mathbf{x} \in F$, by Corollary 3.0.2, we have that:

$$\sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i) \mathbf{u}_i = \mathbf{c} \ \ \forall \mathbf{x} \in F, \quad \text{ i.e. } \sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i)(\mathbf{u}_i)_j = c_j \ \ \forall \mathbf{x} \in F, \ \forall j \in V.$$

¿From the equation of $\sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i)^2 = 1$, we get that:

$$c_j = \sum_{i \in V} x_i c_j (\mathbf{c}^T \mathbf{u}_i)^2 = \sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i)(\mathbf{u}_i)_j.$$

Since F is $n-1$ dimensional, these equations imply that:

$$c_j (\mathbf{c}^T \mathbf{u}_i)^2 = (\mathbf{c}^T \mathbf{u}_i)(\mathbf{u}_i)_j \quad \text{ for all } \ i, j \in V$$

Hence for any $i \in V$ such that $(\mathbf{c}^T \mathbf{u}_i) \neq 0$, we have $(\mathbf{c}^T \mathbf{u}_i)\mathbf{c} = \mathbf{u}_i$. Since $\|\mathbf{u}_i\| = \|\mathbf{c}\| = 1$, this yields that $\mathbf{u}_i = \pm \mathbf{c}$. We may assume w.l.o.g. that $\mathbf{u}_i = \mathbf{c}$.

Thus for each i, either $\mathbf{c}^T \mathbf{u}_i = 0$ or $\mathbf{u}_i = \mathbf{c}$. Let $K = \{i \in V \ : \ \mathbf{u}_i = \mathbf{c}\}$. Then K is a clique, since for any two nodes $i, j \in Q$ we have that $\mathbf{u}_i^T \mathbf{u}_j = \mathbf{c}^T \mathbf{c} = 1 \neq 0$, i.e. i, j are adjacent, because $(\mathbf{u}_i \mid i \in V) \in ONR(G, N)$.

So the inequality of $\sum_{i \in V} x_i (\mathbf{c}^T \mathbf{u}_i)^2 \leq 1$ (which was equivalent to our facet defining inequality) is just the clique inequality for K, and this proves the theorem. ■