Semidefinite Programming
Lecture 10

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We have already seen that

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \text{Q-STAB}(G). \tag{1.1}$$

Plugging in the complement of the graph G, $\overline{G}$ in 1.1, we get

$$\mathrm{Clique}(G) \subseteq \mathrm{TH}(\overline{G}) \subseteq \text{Q-Clique}(\overline{G}) \tag{1.2}$$

where Q-Clique$(\overline{G})$ corresponds to $\{x | 1_s^T x \leq 1,\ \forall s \textit{ stable set in } \overline{G}\}$. Applying *anti-blocker* operator to 1.1 yields to

$$\text{Q-Clique}(G) \supseteq \mathrm{TH}(\overline{G}) \supseteq \mathrm{Clique}(G) \tag{1.3}$$

Now, we can state the following result;

Corollary 1.1. *G is perfect iff $\overline{G}$ is perfect.*

Proof. By definition, G is perfect iff $\mathrm{STAB}(G) = \text{Q-STAB}(G)$, and applying anti-blocker to this realtion gives raise to $\text{Q-Clique}(G) = \mathrm{Clique}(G)$.
By definition $\text{Q-Clique}(G) = \text{Q-STAB}(\overline{G})$ and $\mathrm{Clique}(G) = \mathrm{STAB}(\overline{G})$, which means $\text{Q-STAB}(\overline{G}) = \mathrm{STAB}(\overline{G})$ or equivalently $\overline{G}$ is perfect.□

Recall the following theorem from the previous lecture;

Theorem 1.1. *The inequality $ax \leq \alpha$ is a facet, i.e. $(n-1)$ dimensional face of* $\mathrm{TH}(G)$ *iff $ax \leq \alpha$ is equivalent to either $x_i \geq 0$ or $1_k^T x \leq 1$, for k clique in G.*

Using the above facts, we can state the following result;

Corollary 1.2. $\mathrm{TH}(G)$ *is a polytope iff G is perfect.*

Proof. The necessity is trivial; if G is perfect then $\mathrm{TH}(G) = \mathrm{STAB}(G)$, which implies$\mathrm{TH}(G)$ is a polytope.
Suppose $\mathrm{TH}(G)$ is a polytope, then using Theorem 1.1, (i.e. $\mathrm{TH}(G) \supseteq$ Q-STAB(G)), and the fact that $\mathrm{TH}(G) \subseteq$ Q-STAB(G), we will get $\mathrm{TH}(G) =$ Q-STAB(G). By applying the anti-blocker, we get $\mathrm{TH}(\overline{G}) = \mathrm{STAB}(\overline{G})$. Since $\mathrm{TH}(\overline{G})$ is a polytope, we have $\mathrm{TH}(\overline{G}) =$ Q-STAB$(\overline{G})$, which implies $\overline{G}$ is a perfect graph.Then by Corollary 1.1 G is also perfect.□

Given A graph $G = (V, E)$, denote by $\omega(G)$ and $\alpha(G)$ the sizes of maximum clique and maximum stable sets, respectively.

Definition. *G is ω-perfect if for every induced subgraph G', $\omega(G') = \chi(G')$ where $\chi(G')$ is the chromatic number of G'. Also, G is α-perfect if for any induced subgraph G', $\alpha(G') = \rho(G')$ where $\rho(G') = \chi(\overline{G'})$.*

Next, we will see these definitions are equivalent to our definition of a perfect graph.

THEOREM 1.2. *The following statements are equivalent:*

i. G is perfect (i.e. $\overline{G}$ is perfect).

ii. G is ω-perfect (i.e. $\overline{G}$ is α-perfect).

iii. G is α-perfect (i.e $\overline{G}$ is ω-perfect).

Consider the optimization problems

$$\omega^*(G, w) = \max\{w^T x | 1_s^T x \leq 1, \ \forall s \ stable\ set\} \tag{1.4}$$

and

$$\omega^*(G, w) = \max\{w^T x | x clique\}. \tag{1.5}$$

The dual of 1.4 can be stated as follows

$$\min\{\sum_{s\ stable\ set} y_s | \sum_{s \ni i} y_s \geq w_i \ i = 1, \ldots, n\}. \tag{1.6}$$

Clearly in case of perfect graphs, 1.4 and 1.5 are equivalent. Although if w is an integral vector, the optimum values of 1.4 and 1.5 will be integral, but it does not imply that the optimizer of 1.6 is an integral vector.
Now assume that y_s is an 0-1 vector. The $\sum y_s$ can be interpreted as total number of colors. Also, constraints requirement implies that each vertex i has at least w_i different colors. This means that, in an 0-1 optimal solution, we minimize the number of colors, provided that each vertex has at at least w_i distinct colors. In other words we define the weighted chromatic number of G with respect to integral weight vector w, $\chi(G, w)$, to be the minimum number of colors, so that each vertex i has at least w_i colors.

Proof of Theorem 1.2.

$i \Rightarrow ii$:
The proof is by induction on the number of vertices in G.
Observe that Clique(G) = Q-Clique(G), and in particular if the weight vector belongs to $\{0,1\}^n$, then we only work with a subset of vertices of the graph. So we only pick those vertices with with $w_i = 1$. It also means that we will not pick an edge which either one of its endpoints has weight 0.
So for the case of $w \in \{0,1\}^n$, we are only working with induced subgraphs of G.
Clearly, Clique(G') = Q-Clique(G'). This equality follows from the fact that Clique(G') and Q-Clique(G') are projections of Clique(G) and Q-Clique(G) along those vertices with zero entry in w. It can also be derived from equivalence of optimal values of 1.4 and 1.5 for any integral weight vector. This means that for the case of $w \in 0, 1^n$, we have $\omega(G,w) = \omega(G')$ and $\omega^*(G,w) = \omega^*(G')$. A coloring of vertices that uses precisely $\omega(G)$ colors concludes this part of the proof.
Suppose $i \Rightarrow ii$ is true for all graphs with fewer than $|V|$ vertices, which means this is also true for all induced subgraphs of G. Consider the following optimization problem

$$\begin{array}{ll} \min & 1^T x \\ \text{s.t.} & 1_s^T x \leq 1 \\ & \forall s \text{ stable set in } G. \end{array} \tag{1.7}$$

An integral optimal solution of this problem corresponds to a maximum clique of G, with respect to weight vector $\tilde{w} = (1, \dots, 1)^T$. Define F as the convex hull of all such maximum cliques of G. Clearly F is a face of polytope Clique(G). Observe that there exist a facet of Clique(G) that includes F. Since G is perfect, there exists a stable set s such that $1_s^T x = 1$ for all $x \in F$. This implies that s intersects any maximum clique k. By induction hypothesis, the condition $i \Rightarrow ii$ is satisfied for induced subgraph $G' = G - S$, i.e. there exists a coloring of G' with $\omega(G')$ colors. Clearly $\omega(G') = \omega(G) - 1$. Now adding s back into G', keeping the old coloring of G' and adding a new color for s, we have a valid coloring for G with at most $\chi(G) = \omega(G') + 1 = \omega(G)$ colors.□
$ii \Rightarrow i$:
For a general graph with an arbitrary weight vector, we have the following chain of inequalities

$$\omega(G,w) \leq \omega^*(G,w) \leq \chi^*(G,w) \leq \chi(G,w). \tag{1.8}$$

For ω-perfect graphs, if $w \in \{0,1\}^n$ then $\omega(G,w) = \chi(G,w)$. This is also the case for all vectors $w \geq 0$. Start with the case where w is an integral vector. Extending this result to rational numbers is accomplished by scaling w with common denominator of all entries in it. Then we can extend it to positive reals by using a limiting process.
Now for an integral vector w, $\omega(G,w) = \omega(G_w)$ where G_w is a graph obtained from G by replacing each vertex i with a clique of size w_i, and by connecting all vertices arising from vertex i with all vertices arising from vertex j iff $ij \in E$. Clearly if this is for the case of replicating only one vertex, then the general case also follows.

LEMMA 1.1. *Let x_1 be a new vertex generated by replicating x, a vertex of an ω-perfect graph G. Then the new graph $G_1 = G + x_1$ is still ω-perfect.*

Proof. The proof is by induction on number of vertices of graph. Suppose that Lemma is true for all graphs with fewer than $|V_{G_1}|$ vertices.
The first claim is that all induced subgraphs of G_1 are ω-perfect. If the induced subgraph does not contain both x and x_1, the resulting graph is equal to an induced subgraph of G that we know is ω-perfect. In case the induced subgraph contains both x and x_1, it is just like applying this Lemma to a graph with fewer vertices than $|V_{G_1}|$, which by induction hypothesis is true.□ It remains to show that $\omega(G_1) = \chi(G_1)$. First assume that the replicated vertex is part of some maximum clique, then $\omega(G_1) = \omega(G) + 1$. Also $\chi(G_1)$ will increase by one, since we already have a valid coloring of G with $\chi(G)$ colors and we only need one new color for replicated vertex.
Next assume that x is not part of a maximum clique, i.e. $\omega(G_1) = \omega(G)$. We claim there is a coloring of G, with a color class S which contains x and intersects all the maximum cliques of G. Once we have result the induced subgraph $G_1 - S + x$ can be colored using $\omega(G) - 1$ colors; giving S back and using one more color for S we get a proper coloring by $\omega(G)$ colors. To show the claim, observe that the graph $G - S$ has a proper coloring of size $\omega(G) - 1$, which means $G_1 - S$ also has a proper coloring of size $\omega(G_1) - 1$. Then the coloring class S hits every maximum clique.
This implies that the graph G_w is ω-perfect, and it follows that

$$\omega(G, w) = \chi(G, w). \tag{1.9}$$

or

$$\omega(G, w) = \omega^*(G, w). \tag{1.10}$$

Using anti-blocker we have $ii \Rightarrow i$. Clearly, applying the same argument to complement graph proves the equivalence of i and iii.□