Complementarity and Nondegeneracy in Semidefinite Programming *

Farid Alizadeh[†] Jean-Pierre A. Haeberly[‡] Michael L. Overton[§]

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Abstract

Primal and dual nondegeneracy conditions are defined for semidefinite programming. Given the existence of primal and dual solutions, it is shown that primal nondegeneracy implies a unique dual solution and that dual nondegeneracy implies a unique primal solution. The converses hold if strict complementarity is assumed. Primal and dual nondegeneracy assumptions do not imply strict complementarity, as they do in LP. The primal and dual nondegeneracy assumptions imply a range of possible ranks for primal and dual solutions X and Z. This is in contrast with LP where nondegeneracy assumptions exactly determine the number of variables which are zero. It is shown that primal and dual nondegeneracy and strict complementarity all hold generically. Numerical experiments suggest probability distributions for the ranks of X and Z which are consistent with the nondegeneracy conditions.

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†RUTCOR, Rutgers University, New Brunswick, NJ. Supported in part by NSF postdoctoral grant CDA-9211106. E-mail: `alizadeh@rutcor.rutgers.edu`

‡Mathematics Department, Fordham University, Bronx, NY. Supported in part by National Science Foundation grant CCR-9401119. E-mail:`haeberly@murray.fordham.edu`

§Computer Science Department, Courant Institute of Mathematical Sciences, New York University, New York, NY. Supported in part by National Science Foundation grant CCR-9401136. E-mail: `overton@cs.nyu.edu`

1 Duality and Complementarity

Let $\mathcal{S}^n$ denote the set of real symmetric $n \times n$ matrices. Denote the dimension of this space by

$$n^{\overline{2}} = n(n+1)/2. \tag{1}$$

The standard inner product on $\mathcal{S}^n$ is

$$A \bullet B = \mathbf{tr}\ AB = \sum_{i,j} a_{ij} b_{ij}.$$

By $X \succeq 0$, where $X \in \mathcal{S}^n$, we mean that X is positive semidefinite. The set $\mathcal{K} = \{X \in \mathcal{S}^n : X \succeq 0\}$ is called the positive semidefinite cone. The constraint $X \succeq 0$ is equivalent to a bound constraint on the least eigenvalue of X, which is not a differentiable function of X.

The semidefinite programming problem (SDP) is

$$\begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & A_k \bullet X = b_k \quad k = 1, \ldots, m; \quad X \succeq 0. \end{array} \tag{2}$$

Here C and A_k, $k = 1, \ldots, m$, are all fixed matrices in $\mathcal{S}^n$, and the unknown variable X also lies in $\mathcal{S}^n$. If the constraints are chosen to enforce X to be diagonal one obtains linear programming (LP) as a special case of SDP. The dual of SDP is

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & Z + (\sum_{k=1}^m y_k A_k) = C; \quad Z \succeq 0 \end{array} \tag{3}$$

where Z is a dual slack matrix variable, which also lies in $\mathcal{S}^n$.

As is well known, the SDP primal-dual pair has many of the properties of LP. For feasible X, y, Z the duality gap is $X \bullet Z = \mathbf{tr}\ XZ$, since

$$C \bullet X - b^T y = Z \bullet X + (\sum_{k=1}^m y_k A_k) \bullet X - b^T y = X \bullet Z \geq 0.$$

The following are assumed to hold throughout the paper.

Assumption 1. At least one of the following holds: there exists a primal feasible point $X \succ 0$, or there exists a dual feasible point (y, Z) with $Z \succ 0$.

Assumption 2. There exist finite primal and dual optimal solutions X and (y, Z).

Assumption 3. The matrices $A_k,\ k = 1, \ldots, m$, are linearly independent, i.e. they span an m-dimensional linear space in $\mathcal{S}^n$.

Assumption 1 (the Slater condition) and Assumption 2 imply (see e.g. [NN94]) that the duality gap $X \bullet Z = 0$ for optimal solutions (X, y, Z). As is well known, this implies the complementary condition

$$XZ = 0. \tag{4}$$

To prove this, observe that $X \succeq 0,\ Z \succeq 0$ and $\mathbf{tr}\, XZ = 0$ imply that the matrix $X^{1/2}ZX^{1/2}$ is symmetric, positive semidefinite, and has zero trace. It follows that $X^{1/2}ZX^{1/2} = 0$, and therefore that $XZ = 0$.

The complementarity condition (4) implies that X and Z commute, so they share a common system of eigenvectors. Thus we have:

Lemma 1 *Let X and (y, Z) be respectively primal and dual feasible. Then they are optimal if and only if there exists $Q \in \Re^{n \times n}$, with $Q^TQ = I$, such that*

$$X = Q\ \mathbf{Diag}(\lambda_1, \ldots, \lambda_n)\ Q^T, \tag{5}$$

$$Z = Q\ \mathbf{Diag}(\omega_1, \ldots, \omega_n)\ Q^T, \tag{6}$$

and

$$\lambda_i \omega_i = 0, \quad i = 1, \cdots, n. \tag{7}$$

all hold.

Equation (7) expresses complementarity in terms of the eigenvalues of X and Z. If X has rank r and Z has rank s, complementarity implies $r + s \leq n$. We say that *strict complementarity* holds if $r + s = n$, i.e. for each $i \in \{1, \ldots, n\}$, exactly one of $\lambda_i = 0$ or $\omega_i = 0$ holds.

2 Nondegeneracy and Strict Complementarity

In Section 1 we noted the similarities between SDP and LP, but in this section we shall emphasize the differences. To some extent our discussion is motivated by results for eigenvalue optimization given by Overton and Womersley[OW93, OW95] and Shapiro and Fan[SF95]. Shapiro [Sha95] gives related results and extends these to nonlinear SDP's.

Consider the set

$$\mathcal{M}_k = \{X \in \mathcal{S}^n : \ \mathbf{rank}(X) = k\}.$$

Since the eigenvalues of a matrix X are continuous functions of X, it is clear that, for $k > 0$, the boundary of $\mathcal{M}_k$ is

$$\partial\mathcal{M}_k = \mathcal{M}_0 \cup \cdots \cup \mathcal{M}_{k-1}.$$

Let

$$\mathcal{M}_k^+ = \mathcal{K} \cap \mathcal{M}_k = \{X \in \mathcal{S}^n : \quad X \succeq 0 \text{ and } \mathbf{rank}(X) = k\}.$$

Then the boundary of $\mathcal{K}$ is given by

$$\partial\mathcal{K} = \mathcal{M}_0^+ \cup \cdots \cup \mathcal{M}_{n-1}^+ \tag{8}$$

and the interior of $\mathcal{K}$ is

$$\text{Int } \mathcal{K} = \mathcal{M}_n^+.$$

Before going further, let us consider analogous definitions for the non-negative orthant $\mathcal{J} = \{x \in \Re^n \ : x \geq 0\}$. Consider the set

$$\mathcal{L}_k = \{x \ \in \Re^n \ : \quad x \text{ has exactly } k \text{ nonzero elements}\}.$$

For $k > 0$ the boundary of $\mathcal{L}_k$ is

$$\partial\mathcal{L}_k = \mathcal{L}_0 \cup \cdots \cup \mathcal{L}_{k-1}.$$

Let

$$\mathcal{L}_k^+ = \mathcal{J} \cap \mathcal{L}_k.$$

The boundary of $\mathcal{J}$ is

$$\partial\mathcal{J} = \mathcal{L}_0^+ \cup \cdots \cup \mathcal{L}_{n-1}^+ \tag{9}$$

and the interior of $\mathcal{J}$ is

$$\text{Int } \mathcal{J} = \mathcal{L}_n^+.$$

However, the decompositions (8) and (9) have very different characters. The set $\mathcal{L}_k^+$ is not connected, except in the cases $k = 0$ and $k = n$. For example, for $n = 2$, the set $\mathcal{L}_1^+$ consists of the two positive coordinate axes (excluding the origin). By contrast, the set $\mathcal{M}_k^+$ is a path-connected smooth submanifold of $\mathcal{S}^n$ for all k, $0 \leq k \leq n$. For example, in the case $n = 2$, we have

$$\mathcal{M}_1^+ = \left\{ \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix} : \quad \alpha \geq 0, \ \beta \geq 0, \ \alpha\beta \neq 0, \ \gamma = \pm\sqrt{\alpha\beta} \right\},$$

a connected, smooth submanifold of $\mathcal{S}^2$.

Let X be primal feasible with $\mathbf{rank}(X) = r$ and

$$X = Q \ \mathbf{Diag}(\lambda_1, \ldots, \lambda_r, 0, \cdots, 0) \ Q^T \tag{10}$$

where $Q^T Q = I$. The tangent space to $\mathcal{M}_r$ at X is [Arn71, SF95]

$$\mathcal{T}_X = \left\{ Q \begin{bmatrix} U & V \\ V^T & 0 \end{bmatrix} Q^T \quad : \quad U \in \mathcal{S}^r, V \in \Re^{r \times n-r} \right\}.$$

Recalling the notation (1), $\dim \mathcal{T}_X = r^{\overline{2}} + r(n-r) = n^{\overline{2}} - (n-r)^{\overline{2}}$. For $\Delta X \in \mathcal{T}_X$ we have

$$Q^T (X + \epsilon \Delta X) Q = \begin{bmatrix} \mathbf{Diag}(\lambda_1, \ldots, \lambda_r) + \epsilon U & \epsilon V \\ \epsilon V^T & 0 \end{bmatrix}.$$

Thus $X \pm \epsilon \Delta X$ is *not* contained in $\mathcal{K}$, for $\epsilon > 0$, unless $V = 0$.

Definition 1. X is *primal nondegenerate* if it is primal feasible and

$$\mathcal{T}_X + \mathcal{N} = \mathcal{S}^n, \tag{11}$$

where

$$\mathcal{N} = \{X \in \mathcal{S}^n : \ A_i \bullet X = 0\}. \tag{12}$$

Theorem 1 *Let X be primal feasible with* $\mathbf{rank}(X) = r$. *A necessary condition for X to be primal nondegenerate is that*

$$(n-r)^{\overline{2}} \leq n^{\overline{2}} - m. \tag{13}$$

Furthermore, let $Q_1 \in \Re^{n \times r}$ and $Q_2 \in \Re^{n \times (n-r)}$ respectively denote the first r columns and the last $n-r$ columns of Q given by (10). Then X is primal nondegenerate if and only if the matrices

$$B_k = \begin{bmatrix} Q_1^T A_k Q_1 \\ Q_2^T A_k Q_1 \end{bmatrix}, \ k = 1, \ldots, m \tag{14}$$

are linearly independent. By this we mean that they span an m-dimensional linear space in $\mathcal{S}^r \times \Re^{(n-r) \times r}$, a space whose dimension is $r^{\overline{2}} + r(n-r)$, i.e. $n^{\overline{2}} - (n-r)^{\overline{2}}$.

Proof. Condition (13) follows directly from Definition 1, since $\dim \mathcal{T}_X = n^{\overline{2}} - (n-r)^{\overline{2}}$ and $\dim \mathcal{N} = n^{\overline{2}} - m$. The condition (11) is equivalent to

$$\mathcal{T}_X^\perp \cap \mathcal{N}^\perp = \{0\} \tag{15}$$

where $\mathcal{T}_X^\perp$ and $\mathcal{N}^\perp$ are respectively the orthogonal complements of $\mathcal{T}_X$ and $\mathcal{N}$, namely

$$\mathcal{T}_X^\perp = \left\{ Q \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix} Q^T \ : \qquad W \in \mathcal{S}^{n-r} \right\}$$

and

$$\mathcal{N}^\perp = \mathrm{Span}\{A_k\}.$$

If the B_k are linearly dependent, there exist θ_k not all zero such that $\sum \theta_k B_k = 0$. This contradicts (15), since then $\sum \theta_k A_k \in \mathcal{T}_X^\perp$. If the B_k are linearly independent, (15) holds. □

Note that Theorem 1 holds for any Q satisfying (10).

Theorem 2 *Let X be primal nondegenerate and optimal. Then there exists a unique optimal dual solution (y, Z).*

Proof. By Assumption 2, a dual optimal solution (y, Z) exists, so that complementarity holds. As above, let Q_1 and Q_2 respectively denote the first r columns and the last $n-r$ columns of Q given in (10). Any $\widetilde{Z}$ satisfying the complementarity condition $X\widetilde{Z} = 0$ must be of the form

$$\widetilde{Z} = Q_2 W Q_2^T$$

for some $W \in \mathcal{S}^{n-r}$, so the feasibility condition (3) requires the existence of $\tilde{y} \in \Re^m$ and $W \in \mathcal{S}^{n-r}$ such that

$$Q_2 W Q_2^T + \sum_{k=1}^m \tilde{y}_k A_k = C.$$

The linear independence conditions given by Assumption 3 and Theorem 1 guarantee that any solution of this linear system is unique. □

Note that if we assume Q satisfies (6) as well as (10) we find that $W = \mathbf{Diag}(\omega_{r+1}, \ldots, \omega_n)$.

Now we turn to dual nondegeneracy. Let (y, Z) be dual feasible with $\mathbf{rank}(Z) = s$ and

$$Z = Q\ \mathbf{Diag}(0, \ldots, 0, \omega_{n-s+1}, \ldots, \omega_n)\ Q^T \tag{16}$$

with $Q^TQ = I$. The tangent space to $\mathcal{M}_s^+$ at Z is

$$\mathcal{T}_Z = \left\{ Q \begin{bmatrix} 0 & V \\ V^T & W \end{bmatrix} Q^T \quad : \qquad V \in \Re^{(n-s)\times s},\ W \in \mathcal{S}^s \right\}. \tag{17}$$

We have $\dim(\mathcal{T}_Z) = s^{\overline{2}} + s(n-s) = n^{\overline{2}} - (n-s)^{\overline{2}}$.

Definition 2. The point (y, Z) is *dual nondegenerate* if it is dual feasible and Z satisfies

$$\mathcal{T}_Z + \mathrm{Span}\{A_k\} = \mathcal{S}^n. \tag{18}$$

Theorem 3 *Let (y, Z) be dual feasible with $\mathbf{rank}(Z) = s$. A necessary condition for (y, Z) to be dual nondegenerate is that*

$$(n-s)^{\overline{2}} \leq m. \tag{19}$$

Furthermore, let $\widetilde{Q}_1 \in \Re^{n\times(n-s)}$ and $\widetilde{Q}_2 \in \Re^{n\times s}$ respectively denote the first $n-s$ and the last s columns of Q given by (16). Then (y, Z) is dual nondegenerate if and only if the matrices

$$\widetilde{B}_k = [\widetilde{Q}_1^T A_k \widetilde{Q}_1],\ \ k = 1, \ldots, m \tag{20}$$

span $\mathcal{S}^{n-s}$.

Proof. It is an immediate consequence of the definition.

Note that Theorem 3 holds for any Q satisfying (16).

Theorem 4 *Let (y, Z) be dual nondegenerate and optimal. Then there exists a unique optimal primal solution X.*

Proof. By Assumption 2, a primal optimal solution X exists. As above let $\widetilde{Q}_1$ and $\widetilde{Q}_2$ respectively denote the first $n-s$ columns and the last s columns of Q given by (16). Any $\widetilde{X}$ satisfying the complementarity condition $\widetilde{X}Z = 0$ must be of the form

$$\widetilde{X} = \widetilde{Q}_1 U \widetilde{Q}_1^T$$

for some $U \in \mathcal{S}^{n-s}$. Thus the feasibility condition (2) reduces to

$$\widetilde{Q}_1^T A_k \widetilde{Q}_1 \bullet U = b_k, \quad k = 1, \ldots, m \tag{21}$$

Theorem 3 guarantees that any solution of this linear system is unique. $\square$

Note that if we assume Q satisfies (5) as well as (16) we find that $U = \mathbf{Diag}(\lambda_1, \ldots, \lambda_{n-s})$.

Note also the distinction between the partitionings of Q used in Theorems 1 and 3. The former uses $Q = [Q_1 \;\; Q_2]$ where Q_1 has r columns and the latter uses $Q = [\widetilde{Q}_1 \;\; \widetilde{Q}_2]$ where $\widetilde{Q}_1$ has $n-s$ columns. These partitionings are the same if and only if $r+s=n$, i.e. strict complementarity holds.

It is instructive to compare our SDP nondegeneracy definitions with those of the LP

$$\min \quad c^T x \quad \text{subject to } Ax = b, \; x \geq 0,$$

where $A \in \Re^{m\times n}$. Suppose that r is the number of nonzero primal solution variables x_j, with $x_{r+1} = \cdots = x_n = 0$, and s is the number of corresponding nonzero dual slacks z_j, with $z_1 = \cdots = z_{n-s} = 0$. By complementarity, $r + s \leq n$. Consider the partitionings

$$A = [A_1 \; A_2] \quad \text{and} \quad A = [\widetilde{A}_1 \; \widetilde{A}_2]$$

where A_1 has r columns and $\widetilde{A}_1$ has $n-s$ columns. These partitionings are identical if strict complementarity holds. LP primal nondegeneracy states that the m rows of A_1 must be a linearly independent set in $\Re^r$, which requires $r \geq m$. LP dual nondegeneracy states that the m rows of $\widetilde{A}_1$ should span $\Re^{n-s}$, which requires $s \geq n - m$. Combined with the complementarity condition $r + s \leq n$, these conditions imply $r = m$ and $s = n - m$. Thus in LP, primal and dual nondegeneracy imply strict complementarity. This is *not* the case for SDP.

Example. Let $n = 3, m = 3$, with $b = [1 \; 0 \; 0]^T$,

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has the solution

$$X = \mathbf{Diag}(1,0,0), \quad y = [0 \; 0 \; 0], \quad Z = \mathbf{Diag}(0,0,1).$$

That this solution is valid is easily verified by checking the optimality conditions (2), (3), (4). We have $Q = I$, with the eigenvalues λ_i and ω_i equal to the diagonal elements of X and Z respectively. Note that $r = 1$ and

$s = 1$, so strict complementarity does *not* hold. Let us check the primal nondegeneracy condition. We have

$$Q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so the matrices B_k^T, $k = 1, 2, 3$, defined by (14), are

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Since these are linearly independent, the primal nondegeneracy condition holds, and the dual solution must be unique. Now let us check the dual nondegeneracy condition. We have

$$\widetilde{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the matrices $\widetilde{B}_k$, $k = 1, 2, 3$, defined by (20), are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since these span $\mathcal{S}^2$, the dual nondegeneracy condition holds, and the primal solution must be unique. Note especially that, in this example, strict complementarity fails to hold even in the presence of primal and dual nondegeneracy.

Theorems 2 and 4 show that primal and dual nondegeneracy respectively imply dual and primal unique solutions. The converses are true assuming strict complementarity:

Theorem 5 *Suppose that X and (y, Z) are respectively primal and dual optimal solutions satisfying strict complementarity. Then if the primal solution X is unique, the dual nondegeneracy condition must hold, and if the dual solution (y, Z) is unique, the primal nondegeneracy condition must hold.*

Proof. Let Q satisfy conditions (10) and (16), as in Lemma 1. Strict complementarity states that $r + s = n$, so the partitionings of Q used in Theorems 1 and 3 are the same. Thus

$$X = Q_1 \, \mathbf{Diag}(\lambda_1, \ldots, \lambda_r) \, Q_1^T, \quad Z = Q_2 \, \mathbf{Diag}(\omega_{r+1}, \ldots, \omega_n) \, Q_2^T.$$

Suppose first that the dual nondegeneracy assumption (18) fails to hold. We shall show that in this case X cannot be a unique primal solution. Complementarity states that any solution $\widetilde{X}$ must satisfy

$$\widetilde{X} = Q_1 U Q_1^T$$

for some $U \in \mathcal{S}^r$, and so the primal feasibility condition (2) reduces to

$$Q_1^T A_k Q_1 \bullet U = b_k, \quad k = 1, \ldots, m.$$

Because the dual nondegeneracy assumption does not hold, the solution set of this equation is *not* unique, but holds on an affine subset of $\mathcal{S}^r$, say $\mathcal{U}$, with positive dimension. The condition that $\widetilde{X} \succeq 0$ holds if and only if $U \succeq 0$. But the particular choice $U = \mathbf{Diag}(\lambda_1, \ldots, \lambda_r)$ lies in $\mathcal{U}$ and is positive definite, so there is an open set in $\mathcal{U}$ for which the same is true. Every such U defines an $\widetilde{X}$ which satisfies the optimality conditions.

Now suppose that the primal nondegeneracy assumption (11) fails to hold. We shall show that in this case (y, Z) cannot be a unique dual solution. Complementarity states that any solution $\widetilde{Z}$ must satisfy

$$\widetilde{Z} = Q_2 W Q_2^T$$

for some $W \in \mathcal{S}^s$, and so the dual feasibility condition (3) reduces to the solvability of

$$Q_2 W Q_2^T + \sum_{k=1}^{m} \tilde{y}_k A_k = C.$$

for some $\tilde{y} \in \Re^m$ and $W \in \mathcal{S}^s$. Because the primal nondegeneracy assumption does not hold, the solution set of this equation is *not* unique, but holds on an affine subset of $\mathcal{S}^s \times \Re^m$, say $\mathcal{W}$, with positive dimension. The condition $\widetilde{Z} \succeq 0$ in (3) holds if and only if $W \succeq 0$. But the particular choice $(\tilde{y} = y, W = \mathbf{Diag}(\omega_{r+1}, \ldots, \omega_n))$ lies in $\mathcal{W}$ with W positive definite, so there is an open set in $\mathcal{W}$ for which the same is true. Every such W defines a $\widetilde{Z}$ which satisfies the optimality conditions. $\Box$

If the assumption of strict complementarity is not made, it is possible that the primal solution is unique even if the dual nondegeneracy assumption fails. Consider Example 1, changing it so that

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{B}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and therefore the dual nondegeneracy assumption does not hold. It follows that U is not uniquely defined by (21): we can take

$$U = \begin{bmatrix} 1 & \theta \\ \theta & 0 \end{bmatrix}$$

for any $\theta \in \Re$. However, only $\theta = 0$ gives $U \succeq 0$ and therefore $\widetilde{X} \succeq 0$.

It is convenient to introduce some further notation at this point. Let

$$\sqrt[\bar{2}]{k} = \{\max j : j^{\bar{2}} \leq k, \quad j = 1, 2, \cdots, k\},$$

i.e. $\sqrt[\bar{2}]{k} = \lfloor h \rfloor$ where h is the positive real root of $h^{\bar{2}} = k$. We then have:

Theorem 6 *Suppose that X and (y, Z) are respectively primal and dual nondegenerate and optimal, with* $\mathbf{rank}(X) = r$ *and* $\mathbf{rank}(Z) = s$. *Then*

$$n - \sqrt[\bar{2}]{n^{\bar{2}} - m} \quad \leq \quad r \quad \leq \quad \sqrt[\bar{2}]{m} \tag{22}$$

and

$$n - \sqrt[\bar{2}]{m} \quad \leq \quad s \quad \leq \quad \sqrt[\bar{2}]{n^{\bar{2}} - m}. \tag{23}$$

Proof. The lower bounds in (22), (23) are the necessary conditions (13) and (19) given by Theorems 1 and 3. The upper bounds follow from the complementarity condition $r + s \leq n$. $\square$

The ranges of possible values for the ranks of solutions X and Z stand in contrast with LP, where nondegeneracy assumptions give precise formulas for the number of nonzero primal and dual variables. In fact, (22), (23) reduce to equalities only in the cases $m = 0$ ($r = 0$, $s = n$) and $m = n^{\bar{2}}$ ($r = n$, $s = 0$).

Pataki [Pat94] has shown that there always exist optimal solutions X and Z satisfying the *upper* bounds in (22) and (23). Nondegeneracy assumptions are not required for these results. However, without nondegeneracy assumptions the upper bounds need not hold for all solutions, and the lower bounds may not hold for any solution.

We now compare our nondegeneracy conditions with that given by Anderson and Nash [AN87, p.21] in the context of infinite-dimensional LP over general cones. Let $\mathcal{B}_X$ be the linear span of the face of $\mathcal{K}$ generated by X (the face of $\mathcal{K}$ containing X and having minimal dimension). The Anderson-Nash nondegeneracy condition applied to SDP is

$$\mathcal{B}_X + \mathcal{N} = \mathcal{S}^n. \tag{24}$$

It is well known, e.g.[Tau67, p.182], that

$$\mathcal{B}_X = \left\{ Q \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} Q^T \quad : \quad U \in \mathcal{S}^r \right\}. \tag{25}$$

To prove this, note that for ΔX in (25),

$$Q^T(X + \epsilon \Delta X) Q = \begin{bmatrix} \mathbf{Diag}(\lambda_1, \ldots, \lambda_r) + \epsilon U & 0 \\ 0 & 0 \end{bmatrix}$$

so that $X \pm \epsilon \Delta X \in \mathcal{M}_r^+ \subset \mathcal{K}$ for sufficiently small ϵ. This is not true for $\Delta X \notin \mathcal{B}_X$. We have $\mathcal{B}_X \subset \mathcal{T}_X$, with dim $\mathcal{B}_X = \overline{r^2}$. Likewise

$$\mathcal{B}_Z = \left\{ Q \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix} Q^T \quad : \quad W \in \mathcal{S}^s \right\}$$

and so the Anderson-Nash nondegeneracy condition applied to the dual is

$$\mathcal{B}_Z + \mathrm{Span}\{A_k\} = \mathcal{S}^n. \tag{26}$$

Assumptions (24) and (26) imply that

$$\overline{r^2} \geq m \text{ and } \overline{s^2} \geq \overline{n^2} - m$$

must both hold. However, these inequalities can never hold simultaneously, except in the trivial cases $m = 0$ and $m = \overline{n^2}$, because $r + s \leq n$. The relationship between the Anderson-Nash conditions and ours is clarified by noting that (13) and (19) can be written

$$\overline{r^2} + r(n-r) \geq m \text{ and } \overline{s^2} + s(n-s) \geq \overline{n^2} - m.$$

Anderson and Nash define X to be basic if

$$\mathcal{B}_X \cap \mathcal{N} = \{0\}. \tag{27}$$

They show that a point is basic if and only if it is an extreme point of the feasible set, and, that if an optimal solution exists, there is a basic optimal solution. This provides another way to recover Pataki's results since (27) implies

$$\dim \mathcal{B}_X \leq \overline{n^2} - \dim \mathcal{N}$$

i.e.

$$\overline{r^2} \leq m.$$

We also have:

Theorem 7 *Suppose that X and (y, Z) are respectively primal and dual optimal solutions satisfying strict complementarity. Then X is basic if and only if the dual nondegeneracy condition holds.*

Proof. Since $r + s = n$, we have $\mathcal{B}_X = \mathcal{T}_Z^{\perp}$ (see (25) and (17)). Thus the condition that X be primal basic, namely (27), is equivalent to the condition that (y, Z) be dual nondegenerate, namely (18). $\Box$

3 Generic Properties and Probability Distributions

Primal nondegeneracy, dual nondegeneracy and strict complementarity hold generically, in a sense we shall now describe. A semidefinite program is completely determined by C, $\{A_k\}$ and b; in other words its data is taken from the space

$$\mathcal{S} = \left(\prod_{k=1}^{m+1} \mathcal{S}^n \right) \times \Re^m$$

with dimension $d = \dim\ \mathcal{S} = \overline{n}^2(m + 1) + m$.

Definition 3. Let $\widehat{\mathcal{S}}$ be a (Lebesgue) measurable subset of $\mathcal{S}$ with nonzero measure. A property P is a generic property of semidefinite programs in $\widehat{\mathcal{S}}$ if it fails to hold only on a subset of measure zero of $\widehat{\mathcal{S}}$ ([Hal50]).

In other words, a property is generic in $\widehat{\mathcal{S}}$ if it holds almost everywhere.

Assumption 3 is a generic property in $\mathcal{S}$. Assumption 2 holds on a subset of $\mathcal{S}$ with nonzero measure, say $\widetilde{\mathcal{S}}$. Assumption 1 holds generically on $\widetilde{\mathcal{S}}$. Let $\widehat{\mathcal{S}}$ be the set of semidefinite programs for which Assumptions 1, 2 and 3 hold. Then $\widehat{\mathcal{S}}$ has nonzero measure.

Lemma 2 *Primal nondegeneracy, dual nondegeneracy and strict complementarity each hold generically in $\widehat{\mathcal{S}}$.*

Proof: Because of Theorem 5, it suffices to prove that primal uniqueness, dual uniqueness and strict complementarity each hold generically. For primal uniqueness to be violated it is necessary that C be orthogonal to some face of the primal feasible region. Similarly for dual uniqueness to be violated it is necessary that b be orthogonal to some face of the dual feasible region. Both of these properties are generically false in $\widehat{\mathcal{S}}$.

To complete the proof, we need to show that strict complementarity holds generically. If an SDP has primal and dual solutions X and (y, Z) with $\mathbf{rank}(X) = r$ and $\mathbf{rank}(Z) = s$, then the algebraic system of equations

$$\begin{array}{rcl} A_k \bullet Q\ \mathbf{Diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)\ Q^T &=& b_k, \quad k = 1, \ldots, m \\ \sum_{k=1}^m y_k A_k + Q\ \mathbf{Diag}(0, \ldots, 0, \omega_{n-s+1}, \ldots, \omega_n)\ Q^T &=& C \\ Q^T Q &=& I \end{array},$$

must hold. This is a system of $m + 2\overline{n^2} = m + n^2 + n$ equations in the $r + s + m + n^2$ variables λ_i, ω_i, y_k, Q_{ij}. If $r + s < n$, the property that this system is solvable is generically false in $\widehat{\mathcal{S}}$. □

Now suppose we consider properties of SDP's whose data are distributed according to a given probability distribution, e.g. uniformly in $[0, 1]$, discarding those for which the Assumptions do not hold. We may consider the probability of occurrence of given solution ranks r and s. It follows from Lemma 2 that the probability that r and s satisfy the bounds in (22), (23) is one. A natural question is: what is the probability distribution describing the values that r and s take in the ranges (22), (23)? We shall now show the results of some experiments which address this question. This is a promising area for further theoretical investigation.

Let $n = 10$, and consider m ranging from 5 to 50, since the dimension of the primal variable space is $\overline{n^2} = 55$. We solved 20 different randomly generated problems for each pair r, s, using a primal-dual interior-point method. Tables 1 and 2 show the number of times each rank pair (r, s) was achieved. Table 3 shows the generic possible range for r and s for each m, given by (22), (23). The results are consistent with the fact, proved above, that the nondegeneracy and strict complementarity conditions hold with probability one. The results also show clearly that values of r and s in the center of their ranges are much more likely to occur than values equal to the bounds.

We close by noting that the issues of primal and dual nondegeneracy are fundamental to the analysis of convergence rates of primal-dual interior-point methods for SDP. These issues will be discussed in a forthcoming paper.

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m	0	1	2	3	4	5	6	7	8	9	10
5	0	17	3	0	0	0	0	0	0	0	0
10	0	0	19	1	0	0	0	0	0	0	0
15	0	0	1	18	1	0	0	0	0	0	0
20	0	0	0	6	13	1	0	0	0	0	0
25	0	0	0	0	14	6	0	0	0	0	0
30	0	0	0	0	1	12	7	0	0	0	0
35	0	0	0	0	0	3	16	1	0	0	0
40	0	0	0	0	0	0	7	12	1	0	0
45	0	0	0	0	0	0	0	9	11	0	0
50	0	0	0	0	0	0	0	0	17	3	0

Table 1
Number of Occurrences of Rank(X)
in 20 Randomly Generated Problems with $n = 10$

m	0	1	2	3	4	5	6	7	8	9	10
5	0	0	0	0	0	0	0	0	3	17	0
10	0	0	0	0	0	0	0	1	19	0	0
15	0	0	0	0	0	0	1	17	2	0	0
20	0	0	0	0	0	1	12	7	0	0	0
25	0	0	0	0	0	5	15	0	0	0	0
30	0	0	0	0	5	14	1	0	0	0	0
35	0	0	0	1	16	3	0	0	0	0	0
40	0	0	1	11	8	0	0	0	0	0	0
45	0	0	10	10	0	0	0	0	0	0	0
50	0	3	17	0	0	0	0	0	0	0	0

Table 2
Number of Occurrences of Rank(Z)
in 20 Randomly Generated Problems with $n = 10$

m	Bounds on Rank(X)	Bounds on Rank(Z)
5	$1 \le r \le 2$	$8 \le s \le 9$
10	$1 \le r \le 4$	$6 \le s \le 9$
15	$2 \le r \le 5$	$5 \le s \le 8$
20	$3 \le r \le 5$	$5 \le s \le 7$
25	$3 \le r \le 6$	$4 \le s \le 7$
30	$4 \le r \le 7$	$3 \le s \le 6$
35	$5 \le r \le 7$	$3 \le s \le 5$
40	$5 \le r \le 8$	$2 \le s \le 5$
45	$6 \le r \le 9$	$1 \le s \le 4$
50	$8 \le r \le 9$	$1 \le s \le 2$

Table 3
Generic Bounds on Rank for $n = 10$

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